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An Equilibrium Pricing for OTC Derivatives with Non-Cash Collateralization*

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Abstract

In this study, we propose an equilibrium pricing rule for contingent claims traded in over-the-counter (OTC) markets with non-cash collateralization. Owing to counterparty risks in OTC markets, collateral is required to create a derivative contract. The class of assets used as collateral has recently expanded, while cash has always been used as a collateral. Here, we suppose that the required collateral is not cash, and is instead assets with a senior credit class, such as a US government bond. We further assume that market participants source collateral from the repurchase market (SC repo), where investors can borrow assets. Therefore, we provide an equilibrium pricing model that includes the repo market under counterparty risk. Using our pricing rule, we examine the effects of the repo market on OTC derivative transactions from a microeconomics point of view.

JEL Classification: G10, G12, G13
Keywords: OTC derivative pricing, counterparty risk, collateral, pricing kernel, repo market

1 Introduction

This study proposes an equilibrium pricing model for over-the-counter (OTC) derivatives with a counterparty risk when an asset other than cash is used as collateral. We then analyse the effects of this collateralization on OTC derivatives contracts.

The financial crisis in 2008 made practitioners and researchers realize the existence of counterparty risks in the OTC derivatives market. Counterparty risk is a sort of credit risk in the OTC derivatives market, that is, the possibility that the liability side of the derivative contract fails to pay the asset side. To avoid a loss due to the counterparty risk, the asset side can require that the liability side post collateral (i.e. collateralization). Other methods of managing the counterparty risk have been proposed and implemented, such as hedging using credit value-added (CVA), novation, and using central counterparties (CCPs) (Gregory 2010). In fact, counterparty risk is reduced by using a CCP (Duffie and Zhu 2011). Acharya and Bisin (2014) show that changing from an

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OTC transaction to a CCP transaction can prevent the contagion of counterparty risk[1]. In this study, we focus on a collateral agreement as a counterparty risk management method.

Collateralization is a traditional tool to mitigate credit risks in financial markets. Several researchers have investigated the effects of collateralization on asset markets based on microeconomic analyses. Geanakoplos (1996) shows that the introduction of a collateral agreement reduces the traded volume of an asset and improves Pareto efficiency, which decreases with the existence of a default. Acharya and Bisin (2014) demonstrate that an increase in collateral decreases the supply of an insurance product. Takino (2016b) shows that an increase in collateral wholly decreases the liquidity of an OTC option. That is, there is a trade-off between mitigating counterparty exposure and liquidity. In the above studies, cash is used as a collateral asset or the asset class of collateral is not specified. However, other securities have also been used as collateral, including government securities, corporate bonds, and so on (c.f. Chapter 3 in Gregory 2010).

Here, we consider the case in which a government bond with a senior credit rating is used as collateral. In general, senior-rated government bonds are regarded as being safe from default. However, even if the default risk of the collateral asset is removed, the bond price changes stochastically. Hence, it might be risky for the asset side to receive the non-cash asset as collateral when the liability side defaults. On the other hand, the liability side should source the bond as collateral for the counterparty. In practice, such a security is funded in a repurchase (repo) market, where the market participants borrow the security by posting cash to the value of the security as collateral for lenders. The borrowed securities in the repo market are referred as 'special', and a repo market in which such securities are traded is called 'SC repo market'. The agent borrowing the asset in the SC repo market returns it at the final date and then recovers the posted money, including interest (SC repo rate). As defined in Huh and Infante (2016), the SC repo rate is less than the risk-free rate. Denoting the risk-free rate and the SC repo rate as r and r_s, respectively, the difference $r - r_s (> 0)$ is regarded as the cost of funding collateral, per unit of collateral. Furthermore, the value of posted collateral is discounted (i.e. haircut) according to the collateral agreement. The liability side is required greater collateral when the haircut is applied to evaluate collateral. Lou (2017) developed an option pricing rule called liability-side pricing, and derived a Black-Scholes partial differential equation (BS-PDE) that satisfies the option price. Lou (2017) considers a situation in which a non-cash asset is used as collateral, and shows that the discount rate in the Black-Scholes (BS) formula is not only determined by the risk-free interest rate, but also by the collateral asset price, the haircut, and the SC repo rate.

We consider an OTC derivative contract with non-cash collateralization. For convenience, we suppose that the counterparty risk occurs unilaterally. That is, the seller of the claim only risks possible default up to the maturity date of the derivative. Thus, the buyer and the seller refer to the asset side and the liability side, respectively. We consider (static) utility maximization problems for participants' final wealth. By using mean-variance utility, we can explicitly represent the demand and supply functions for the derivative. We derive the equilibrium price and volume for the claim from the market equilibrium. Here, we suppose that the investors determine the claim volumes to purchase or sell after realizing the collateral amount. The collateral value is measured using the marked-to-market (MtM) value of the claim without counterparty risk, and the MtM value is evaluated using risk-neutral pricing. In order to ensure consistency with our equilibrium pricing formula, we use a pricing kernel as a risk-neutral pricing approach. We obtain the pricing kernel by incorporating the optimal behaviors of dealers in the SC repo market.

From the equilibrium formulae, we find the relation between the SC repo rate and the equilibrium. That is, the increase in the SC repo rate increases the supply of the claim, so that the

[1] They use the term of 'counterparty risk externality'.

equilibrium price decreases and the equilibrium volume increases. The increase in the SC repo rate implies an improvement in the funding asset cost. This leads to an increase in the supply. Other results are obtained by numerical examination. We consider a continuous time model. That is, stochastic processes such as the asset price process are described using stochastic differential equations. We also consider a stochastic volatility model because our method enables us to incorporate incomplete market models. This is one of our contributions. Furthermore, we use a reduced-form model to describe the counterparty risk, enabling us to account for a wrong-way risk by setting a correlation parameter between the underlying asset price and the default intensity.

In the numerical examination, we first observe the effects of non-cash collateralization on the equilibrium price and volume of the derivative by comparing it to the cash collateralization case. The results are as follows. The price under non-cash collateralization is slightly larger than that in the cash collateralization case. The volume under the non-cash collateralization is smaller than that in the cash collateralization case. These differences arise only from the change in the gradient of the supply curve. This implies that non-cash collateralization has a greater effect on the liability side than it does on the asset side when non-cash security is used as collateral and both sides have no hedging position on the market risk of the collateral asset. Next, we compare the prices from our model to those from a risk-neutral pricing formula. The risk-neutral formula is taken from Lou (2017). That is, we use the SC repo rate as a discount rate in our risk-neutral formula. The results show that the difference between our equilibrium price and the risk-neutral price is not large for each collateral amount.

The remainder of the article is organized as follows. In the next section, we introduce a financial market model that includes the collateral agreement. In addition, we define the wealth of the market participants. In Section 3, we determine a pricing kernel that enables us to use risk-neutral pricing. In Section 4, we provide the equilibrium for the non-cash collateralization case after deriving the demand and supply functions. In Section 5, we provide the equilibrium for the cash collateralization case, as in Section 4. In Section 6, we numerically implement our equilibrium formulae using Monte-Carlo simulations. The results in this section describe the main results of this work. Section 7 concludes the paper.

2 Model and Collateralization

2.1 Financial Market and Counterparty Risk

We consider a probability space $(\Omega, P, \mathcal{F})$. In our economy, there is a bank account with a constant interest rate r (called a risk-free asset), two risky assets S (typically, stock) and B, and a defaultable claim H written on the risky asset S. The prices at time t of risky assets are denoted by S_t and B_t, and the payoff function of the claim is given by

$$H(T) := H(T, S_T).$$

Risky asset B is used as collateral for derivative contracts. For example, government bonds with a senior credit quality are posted to the counterparties as a collateral asset. We suppose that asset B is a sort of default-free government bond with a zero-coupon. We further assume that the maturity of the bond is after than the maturity of the claim. Then, the market participants are exposed to the market risk of the bond price at the claim maturity.

The counterparty risk in the derivative contract is the possibility that the participant fails to provide the full payout of the claim. We suppose that the derivative has a unilateral counterparty risk; that is, the long-holder (short-holder) has positive (negative) exposure in the derivative contract at any time. In the numerical example, we consider a European option contract as an example. We denote the default event of participant i by 1_{D_i}. The default event has been modeled using a reduced form (Takino 2015) and a structural form (Henderson and Liang 2016). At this stage, we do not specify the default model. The short-holder might default before the maturity date of the claim. We suppose that the default payment is made at maturity, even if the default has occurred before maturity.

There are three types of market participants in our economy, namely, the long- and short-holder of the derivative and the agent, who only provides assets in the SC repo markets. We call the agent supplying the asset in the repo market dealer. We denote the long-holder by $j \in \mathcal{M}_l$, the short-holder by $i \in \mathcal{M}_s$, and the dealer in the SC repo market by $m \in \mathcal{M}_d$, where $\mathcal{M}_l$, $\mathcal{M}_s$, and $\mathcal{M}_d$ are sets of long-holders, short-holders, and dealers, respectively.

Next, we set the behavior of the market participants. The investors in the derivative market invest their initial wealth X_0^h ($h \in \mathcal{M}_l \cup \mathcal{M}_s$) in a risk-free asset, a risky asset, and a derivative with counterparty risk. The long-holder of the derivative receives non-cash collateral from the short-holder owing to her/his counterparty risk, and she/he holds the posted asset until the maturity date of the derivative. The long-holder has to return the collateral if the short-holder does not default. Note that the value of the collateral at maturity might exceed the loss of default. However, we suppose that the long-holder does not return the amount of this difference, for convenience. The short-holder posting the non-cash collateral to the long-holder is assumed to have no assets to use as collateral, and, thus, sources assets from the so-called special collateral repo (SC repo) market by posting cash as collateral in the SC repo market. The short-holder obtains the cash needed for the repo transaction by shorting the asset. The posted cash is returned with interest, which accrues at a rate called the SC repo rate, denoted by r_s, and we assume that r_s is a constant and $r_s < r$, without loss of generality. The participants supplying the asset to the SC repo market purchase the asset in the asset market.

2.2 Collateral Agreement and Payoff with Collateralization

In order to hedge against counterparty risk, the investor with the positive exposure could receive the non-cash collateral from the counterparty with the negative exposure. The value of the collateral per unit of claim is determined using the mark-to-market (MtM) value. We denote MtM value at time t by V_t per unit of claim. We suppose that the MtM value is evaluated through risk-neutral pricing, and the MtM is executed at the contract date (i.e. $t = 0$) only; that is, the collateral is posted as an initial margin. In addition, we introduce a coverage ratio ϕ (≥ 0) as in Fujii and Takahashi (2013). Then, the value of collateral $C(t;\phi)$ per unit of claim with coverage ratio ϕ at time t is

$$C(t) := C(t;\phi) = \phi V_t. \tag{2.1}$$

The coverage ratio ϕ is used to adjust the amount of collateral in this study. We consider non-cash collateral. That is, the agent with the positive exposure receives assets as collateral from her/his counterparty. We assume that the agent with the negative exposure initially posts the collateral assets worth $C(t)$ per unit of derivative contract. That is, the value of the collateral asset actually posted by the investor with the negative exposure per unit of claim is

$$\frac{C(0)}{B_0} B_0 = C(0),$$

where B_0 is the price of the collateral asset at time 0. When n units of derivative contract have been entered, collateral with value $nC(0)$ is posted.

Note that the participants have no full-collateralization transaction when $\phi < 1$. For the non-cash collateralization case, this arises when the haircut is applied instead of introducing the coverage ratio. Lou (2017) states that the liability side should raise funds to the value of the deficit amount in the money market and post it the counterparty or the custodian when the derivative contract is not fully collateralized. This makes the liability side costly. In this study, we suppose that the agent with the negative exposure does not also post cash, even if the contract is not fully collateralized.

Next, we formulate the derivative payoff with the non-cash collateral. We suppose that recovery is zero, for convenience. Then, the long-holder receives $H(T)$ if the short-holder does not default, and the long-holder does not return the posted collateral when the participant defaults (i.e. the long-holder can obtain the asset worth $\frac{C(0)}{B_0}B_T$ per unit of claim). We assume that the derivative payoff with collateral is settled at maturity, even if default occurs before maturity. Therefore, the payoff function of the derivative with non-cash collateralization is

$$g(T) = H(T)(1 - 1_{D_i}) + \frac{C(0)}{B_0} B_T 1_{D_i}, \tag{2.2}$$

where 1_{D_i} is the default indicator function of short-holder i. The first term of (2.2) is the payoff of the claim when short-holder i survives until maturity and the second term is the default payment with collateral when participant i defaults.

2.3 Participants' Total Wealth

The equilibrium price of a claim is determined by its demand-supply equilibrium. The demand and supply functions for the claim are obtained by solving utility maximization problems for each of the participants from their terminal wealth, including the claim. To this end, we introduce wealth equations for each of the market participants.

We first consider the case of the long-holder. Long-holder j ($\in \mathcal{M}_l$) has initial wealth x_0^j which is allocated to the risky asset and the derivative contract. The rest of the money is deposited in the bank account, with a constant interest rate r. The amount deposited in the risky asset by agent h ($\in \mathcal{M}_l \cup \mathcal{M}_s$) is denoted by π_h. We assume that π_h is determined exogenously and is constant for $[0, T]$. The volume or position of the claim that participant $h \in \mathcal{M}_l \cup \mathcal{M}_s$ is willing to trade is denoted by k_h, where $k_j \geq 0$ for $j \in \mathcal{M}_l$ (who are on the buy side), and $k_i \leq 0$ for $i \in \mathcal{M}_s$ (who are on the sell side). Denoting p as the price of the claim at the contract date, the amount deposited in the risk-free asset for long-holder j ($j \in \mathcal{M}_l$) is

$$M_0^j = x_j - \pi_j - k_j p.$$

Then, the terminal wealth is given by

$$X_T^j = M_0^j e^{rT} + \frac{\pi_j}{S_0} S_T + k_j H(T)(1 - 1_{D_i}) - k_j \frac{C(0)}{B_0} B_T (1 - 1_{D_i}) + k_j \frac{C(0)}{B_0} B_T. \tag{2.3}$$

In (2.3), the first term is the value of the bank account, the second term is the value of the risky asset, the third term is the payoff of the defaultable claim sold by agent i ($i \in \mathcal{M}_s$), the fourth is the value of the collateral returned to counterparty i ($i \in \mathcal{M}_s$) if she/he does not default, and the last term is the value of the asset posted from the short-holder as collateral. Then, (2.3) is rewritten as

$$X_T^j = (x_j - \pi_j - k_j p)e^{rT} + \frac{\pi_j}{S_0} S_T + k_j g(T), \tag{2.4}$$

for $j \in \mathcal{M}_l$.

Next, we consider the case of the short-holder. Short-holder i ($i \in \mathcal{M}_s$) has initial wealth x_0^i, and invests this in the risky asset. Since the short-holder has a negative exposure, she/he should post the asset to the buyer as a collateral. The collateral asset is sourced from the SC repo market. The cash deposited in the SC repo market is obtained by short-selling the asset. The rest of the money is deposited in the bank account, with a constant interest rate r. Then, the amount deposited in the risk-free asset for short-holder i ($i \in \mathcal{M}_s$) is

$$M_0^i = x_i - \pi_i + k_i p + k_i C(0) - k_i C(0). \tag{2.5}$$

In (2.5), the fourth term is a short-position of $k_i \frac{C(0)}{B_0}$ units of the asset and the last term denotes the cash collateral posted in the SC repo market to borrow the asset. Then, (2.5) is written as

$$M_0^i = x_i - \pi_i + k_i p.$$

The terminal wealth is given by

$$\begin{aligned} X_T^i =& M_0^i e^{rT} + \frac{\pi_i}{S_0} S_T - k_i H(T)(1 - 1_{D_i}) + k_i \frac{C(0)}{B_0} B_T (1 - 1_{D_i}) \\ &- k_i \frac{C(0)}{B_0} B_T + k_i C(0) e^{r_s T} - k_i \frac{C(0)}{B_0} B_T. \end{aligned} \tag{2.6}$$

In (2.6), the first term is the value of the bank account, the second term is the value of the risky asset position, the third term is the amount of the payout for the defaultable claim, the fourth is the value of the collateral returned from counterparty j ($j \in \mathcal{M}_l$) if the default of agent i does not occur, the fifth is the value to unwind the short-position of the asset, the sixth is the money amount obtained from the SC repo contract, and the last term is the value of the asset returned in the SC repo contract. Then, (2.3) is written as

$$X_T^i = (x_i - \pi_i + k_i p) e^{rT} + \frac{\pi_i}{S_0} S_T - k_i g(T) + k_i \left(C(0) e^{r_s T} - \frac{C(0)}{B_0} B_T \right), \tag{2.7}$$

for $i \in \mathcal{M}_s$.

Finally, we consider the case of the dealer. We assume that the dealer has no asset B supplying the SC repo market, sources assets from the asset market, and supplies those assets to the SC repo market. Denoting the amount of asset B purchased by dealer m by k_m^s ($m \in \mathcal{M}_d$), the money amount deposited in the bank account is

$$M_0^m = x_m + k_m^s B_0 - k_m^s B_0 = x_m.$$

The terminal wealth of dealer m is then

$$X_T^m = x_m e^{rT} - k_m^s B_0 e^{r_s T} + k_m^s B_T. \tag{2.8}$$

Here, by the economic premium principle (c.f. Bühlmann 1980), the unit price p of claim g is given by

$$p = E[\mathcal{E}(T) g(T)], \tag{2.9}$$

and the price B_0 of the asset traded in the SC repo market is given by

$$B_0 = E[\mathcal{E}^s(T) B_T], \tag{2.10}$$

where $\mathcal{E}(T)$ and $\mathcal{E}^s(T)$ are a pricing kernel for claim g and a pricing kernel for asset B, respectively. The pricing kernel is also called state price density, and provides a risk-neutral pricing formula for securities. The pricing kernels are determined through the market equilibrium in the next section. From (2.2), (2.9) is decomposed to

$$E[\mathcal{E}(T)g(T)] = E[\mathcal{E}(T)H(T)] - E[\mathcal{E}(T)H(T)1_{D_i}] + \frac{C(0)}{B_0}E[\mathcal{E}(T)B_T 1_{D_i}].$$

The first term of this equation means that the risk-neutral price of claim H, without the counterparty risk, is given by pricing kernel $\mathcal{E}$. As assumed in the previous section, the MtM value of derivative H is determined through risk-neutral pricing. Therefore, the MtM value at the contract date is given by

$$V_0 = E\left[\mathcal{E}(T)H(T)\right].$$

In the next section, we derive pricing kernel $\mathcal{E}$ from the equilibria of all the financial markets. Therefore, the evaluation of the MtM value (and collateral value) using risk-neutral pricing is consistent with our equilibrium pricing formula for the derivative contract.

3 Pricing Kernel

In this section, we provide pricing kernels used in (2.9) and (2.10) based on the economic premium principle. This principle is a method used to determine the pricing from the market equilibrium, as proposed in Bühlmann (1980), and used in several other studies (Iwaki et al. 2001, Iwaki 2002, Kijima et al. 2010, Takino 2016a, Takino 2017).

We assume that the preference of market participant $h \in \mathcal{M}_l \cup \mathcal{M}_s \cup \mathcal{M}_d$ for risk is represented by an exponential utility function with the risk-averse coefficient γ_h; that is,

$$U_h(x) = 1 - \frac{1}{\gamma_h}e^{-\gamma_h x},$$

for $x > 0$. We denote the inverse function of U'_h by I_h; that is,

$$I_h(x) = (U')^{-1}(x).$$

Agent $h \in \mathcal{M}_l \cup \mathcal{M}_s$ determines the position k_h of the derivative to maximize her/his expected utility from terminal wealth. Then, the objective of participant h ($h \in \mathcal{M}_l \cup \mathcal{M}_s$) is given by

$$E[U_h(X_T^h)] \longrightarrow \text{maximize w.r.t. } k_h.$$

On the other hand, dealer m ($m \in \mathcal{M}_d$) determines the amount of the asset to supply in the SC repo market to maximize her/his expected utility from terminal wealth; that is,

$$E[U_m(X_T^m)] \longrightarrow \text{maximize w.r.t. } k_m^s.$$

Definition 3.1. *The market equilibrium is represented by the following conditions:*

1. $\sum_{i\in\mathcal{M}_s} k_i C(0) + \sum_{m\in\mathcal{M}_d} k_m^s = 0$ *(market-clearing condition of the SC repo market)*
2. $\sum_{j\in\mathcal{M}_l} k_j + \sum_{i\in\mathcal{M}_s} k_i = 0$ *(market-clearing condition of the derivatives market).*

To derive the pricing kernel, we define

$$R_0 := \sum_{h\in\mathcal{M}_l\cup\mathcal{M}_s\cup\mathcal{M}_d}(x_h-\pi_h),$$
$$R_T := \sum_{h\in\mathcal{M}_l\cup\mathcal{M}_s}\frac{\pi_h}{S_0}S_T.$$

Using Definition 3.1, we provide pricing kernels for assets.

Theorem 3.1. *We suppose that our market satisfies the above assumptions and Definition 3.1. In equilibrium, the pricing kernel $\mathcal{E}$ for claim g is given by*

$$\mathcal{E}(T)=\frac{e^{-\gamma R_T}}{e^{rT}E[e^{-\gamma R_T}]}, \tag{3.1}$$

where $\frac{1}{\gamma}=\sum_{h\in\mathcal{M}_l\cup\mathcal{M}_s}\frac{1}{\gamma_h}$. Also, pricing kernel $\mathcal{E}$ for asset B traded in the SC repo market is given by

$$\mathcal{E}^s(T)=M(T)\mathcal{E}(T), \tag{3.2}$$

where $M(T)=e^{(r-r_s)T}$.

Proof. We first consider the problem for long-holder $j\in\mathcal{M}_l$. The first-order condition (FOC) with respect to k_j is

$$E[U_j'(X_T^j)(-pe^{rT}+g(T))]=0.$$

From this, we have

$$p=E\left[\frac{U_j'(X_T^j)}{e^{rT}E[U_j'(X_T^j)]}g(T)\right].$$

By the economic premium principle (2.9), we deduce that

$$\mathcal{E}(t)=\frac{U_j'(X_T^j)}{e^{rT}E[U_j'(X_T^j)]}=:\frac{U_j'(X_j(T))}{L_j}, \tag{3.3}$$

for $j\in\mathcal{M}_l$, where L_j is a constant. From (3.3), it holds that

$$X_T^j=I_j(L_j\mathcal{E}(T)), \tag{3.4}$$

for $j\in\mathcal{M}_l$.

Next, we consider the case of the short-holder. The FOC with respect to k_i ($i\in\mathcal{M}_s$) is

$$E\left[U_i'(X_T^i)\left(pe^{rT}-g(T)+C(0)e^{r_sT}-\frac{C(0)}{B_0}B_T\right)\right]=0.$$

From this, we have

$$p=E\left[\frac{U_i'(X_T^i)}{e^{rT}E[U_i'(X_T^i)]}g(T)\right],$$

and

$$B_0=E\left[\frac{U_i'(X_T^i)}{e^{r_sT}E[U_i'(X_T^i)]}B_T\right].$$

From (2.9) and (2.10), we deduce that

$$\mathcal{E}(T) = \frac{U_i'(X_T^i)}{e^{rT}E[U_i'(X_T^i)]} =: \frac{U_i'(X_T^i(\omega))}{L_i}, \tag{3.5}$$

$$\mathcal{E}^s(T) = \frac{U_i'(X_T^i)}{e^{r_sT}E[U_i'(X_T^i)]}, \tag{3.6}$$

for $i \in \mathcal{M}_s$, where L_i is a constant. From (3.5) and (3.6), we have

$$\mathcal{E}^s(T,\omega) = M(T)\mathcal{E}(T,\omega), \tag{3.7}$$

for all $\omega \in \Omega$, where $M(T) = e^{(r-r_s)T}$. In addition, by (3.5), it holds that

$$X_T^i = I_i(L_i\mathcal{E}(T)), \tag{3.8}$$

for $i \in \mathcal{M}_s$.

As in the case of the dealer, the FOC yields

$$E[U_m'(X_T^m)(-C(0)B_0e^{r_sT} + C(0)B_T)] = 0.$$

From this, we have

$$B_0 = E\left[\frac{U_i'(X_T^m)}{e^{r_sT}E[U_m'(X_T^m)]}B_T\right],$$

for $m \in \mathcal{M}_d$. From (2.10), we deduce that

$$\mathcal{E}^s(t) = \frac{U_m'(X_T^m)}{e^{r_sT}E[U_m'(X_T^m)]} =: \frac{U_m'(X_m(T))}{L_m}, \tag{3.9}$$

where L_m is a constant. From (3.9), it holds that

$$X_T^m = I_m(L_m\mathcal{E}^s(T)), \tag{3.10}$$

for $m \in \mathcal{M}_d$. From (3.7), (3.10) is rewritten to

$$X_T^m = I_m(L_mM(T)\mathcal{E}(T)). \tag{3.11}$$

From Definition 3.1, in the market equilibrium, and summing (3.4), (3.8), and (3.11) for all $j \in \mathcal{M}_l$, $i \in \mathcal{M}_s$, and $m \in \mathcal{M}_d$, respectively, yields

$$R_0e^{rT} + R_T = \sum_{h \in \mathcal{M}_l \cup \mathcal{M}_s \cup \mathcal{M}_d} I_h(L_h\mathcal{E}(T)). \tag{3.12}$$

For the exponential utility case defined above, the inverse function I_h is

$$I_h(x) = -\frac{1}{\gamma_h}\ln x.$$

Then, (3.12) is then rewritten as

$$\frac{1}{\gamma}\ln\mathcal{E}(T) = \bar{L} - R_T, \tag{3.13}$$

where $\frac{1}{\gamma} = \sum_{h \in \mathcal{M}_l \cup \mathcal{M}_s \cup \mathcal{M}_d} \frac{1}{\gamma_h}$ and $\bar{L}$ are constants. Thus, we have

$$\mathcal{E}(T) = e^{\gamma(\bar{L} - R_T)}. \tag{3.14}$$

Taking the expectation of both sides of (3.14) gives

$$E[\mathcal{E}(T)] = e^{\gamma \bar{L}} E[e^{-\gamma R_T}].$$

Since $E[\mathcal{E}(T)] = e^{-rT}$, the constant $\bar{L}$ is given by

$$\bar{L} = \frac{1}{\gamma} \ln \frac{1}{e^{rT} E[e^{-\gamma R_T}]}.$$

Substituting this into (3.14) leads to (3.2). □

4 Demand/Supply Function and Equilibrium

In the previous section, we considered a market equilibrium and derived pricing kernels. In this section, we provide the explicit forms of the demand and supply functions for the derivative contract. To this end, we suppose that there are two participants (i.e. a long-holder and a short-holder) in the derivatives market, and all market participants have a mean-variance criterion as an expected utility (Bessembinder and Lemmon 2002, Acharya and Bisin 2014, Huh and Infante 2016, Takino 2016b/2017). We denote the long-holder by l and the short-holder by s.

The expected utility for participant $h \in \{l, s\}$ is

$$E[U_h(X_T^h)] = E[X_T^h] - \frac{\gamma_h}{2} Var[X_T^h],$$

where γ_h is a risk-aversion coefficient. Recall that the terminal wealth of the long-holder is

$$X_T^l = (x_l - \pi_l - k_l p)e^{rT} + \frac{\pi_l}{S_0} S_T + k_l g(T),$$

and the terminal wealth of the short-holder is

$$X_T^s = (x_s - \pi_s + k_s p)e^{rT} + \frac{\pi_s}{S_0} S_T - k_s g(T) + k_s \left(C(0)e^{r_s T} - \frac{C(0)}{B_0} B_T \right).$$

Thus, the expected utilities are

$$\begin{aligned} E[U^l(X_T^l)] =& (x_l - \pi_l - k_l p)e^{rT} + \frac{\pi_l}{S_0} E[S_T] + k_l E[g(T)] \\ & - \frac{\gamma_l}{2} \left\{ \left(\frac{\pi_l}{S_0} \right)^2 Var[S_T] + k_l^2 Var[g(T)] + 2 \frac{\pi_l}{S_0} k_l Cov[S_T, g(T)] \right\}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} E[U^s(X_T^s)] = & (x_s - \pi_s + k_s p)e^{rT} + \frac{\pi_s}{S_0} E[S_T] - k_s E[g(T)] + k_s \frac{C(0)}{B_0} (B_0 e^{r_s T} - E[B_T]) \\ & - \frac{\gamma_s}{2} \left\{ \left(\frac{\pi_s}{S_0} \right)^2 Var[S_T] + k_s^2 Var[g(T)] + k_s^2 \left(\frac{C(0)}{B_0} \right)^2 Var[B_T] \right. \\ & \left. - 2 \frac{\pi_s}{S_0} k_s Cov[S_T, g(T)] - 2 \frac{\pi_s}{S_0} k_s \frac{C(0)}{B_0} Cov[S_T, B_T] + 2 k_s^2 \frac{C(0)}{B_0} Cov[g(T), B_T] \right\}, \end{aligned} \tag{4.2}$$

respectively.

4.1 Demand Function

The demand function for the claim is given by solving the long-holder's utility maximization problem:

$$\max_{k_l} E[U_l(X_T^l)].$$

For the expected utility (4.1), the FOC yields

$$-pe^{rT} + E[g(T)] - \gamma_l k_l Var[g(T)] - \gamma_l \frac{\pi_l}{S_0} Cov[S_T, g(T)] = 0.$$

Hence, we have the demand function,

$$k_l = \frac{1}{\gamma_l Var[g(T)]} \left(-pe^{rT} + E[g(T)] - \gamma_l \frac{\pi_l}{S_0} Cov[S_T, g(T)] \right). \tag{4.3}$$

4.2 Supply Function

The supply function for the claim is given by solving the short-holder's utility maximization problem:

$$\max_{k_s} E[U_s(X_T^s)].$$

For the expected utility (4.2), the FOC yields

$$\begin{aligned} &pe^{rT} - E[g(T)] + \frac{C(0)}{B_0}(B_0 e^{r_s T} - E[B_T]) \\ &\qquad - \gamma_s k_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_s \frac{\pi_s}{S_0} Cov\left[S_T, g(T) + \frac{C(0)}{B_0} B_T\right] = 0. \end{aligned}$$

Hence, we have the supply function,

$$k_s = \frac{1}{\gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right]} \left(pe^{rT} - E[g(T)] + \frac{C(0)}{B_0}(B_0 e^{r_s T} - E[B_T]) + \gamma_s \frac{\pi_s}{S_0} Cov\left[S_T, g(T) + \frac{C(0)}{B_0} B_T\right] \right). \tag{4.4}$$

4.3 Equilibria

We have obtained the demand and supply functions. From the market-clearing condition

$$k_l = k_s,$$

we derive the equilibrium price and volume for the derivatives contract. Substituting (4.3) and (4.4) into equilibrium relation $k_l = k_s$, we have the equilibrium price p^* of the claim,

$$\begin{aligned} p^* = &\frac{1}{e^{rT}} E[g(T)] - \frac{1}{e^{rT}\left(\gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_l Var[g(T)] \right)} \\ &\times \left\{ \gamma_l \gamma_s \frac{\pi_l}{S_0} Cov[S_T, g(T)] Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_l \gamma_s \frac{\pi_s}{S_0} Cov\left[S_T, g(T) + \frac{C(0)}{B_0} B_T\right] Var[g(T)] \right. \\ &\qquad \left. + \gamma_l Var[g(T)] \frac{C(0)}{B_0}(B_0 e^{r_s T} - E[B_T]) \right\}. \end{aligned} \tag{4.5}$$

Next, substituting (4.5) into (4.3) or (4.4) gives us the equilibrium volume k^* of the claim,

$$k^* = \frac{1}{\gamma_l Var[g(T)]}\left[-\gamma_l \frac{\pi_l}{S_0} Cov[S_T, g(T)] + \frac{1}{\gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_l Var[g(T)]}\right.$$
$$\times \left\{\gamma_l \gamma_s \frac{\pi_l}{S_0} Cov[S_T, g(T)] Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_l \gamma_s \frac{\pi_s}{S_0} Cov\left[S_T, g(T) + \frac{C(0)}{B_0} B_T\right] Var[g(T)]\right.$$
$$\left.\left. + \gamma_l Var[g(T)] \frac{C(0)}{B_0}(B_0 e^{r_s T} - E[B_T])\right\}\right]. \quad (4.6)$$

Although these equilibrium formulae are complex, we can find relations between the equilibria and the SC repo market. From (4.5), and given that $e^{rT}\left(\gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_l Var[g(T)]\right)$ and $\gamma_l Var[g(T)]\frac{C(0)}{B_0}$ are positive, if other variables are unchanged, it holds that

$$\left.\frac{\partial p^*}{\partial r_s}\right|_{B_0 = \text{const.}} < 0.$$

Thus, an increase in the SC repo rate decreases the equilibrium price. From (4.6), and given that $\gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] + \gamma_l Var[g(T)]$ and $\gamma_l Var[g(T)]\frac{C(0)}{B_0}$ are positive, if other variables are unchanged, it holds that

$$\left.\frac{\partial k^*}{\partial r_s}\right|_{B_0 = \text{const.}} > 0.$$

Hence, an increase in the SC repo rate increases the equilibrium price.

These results are explained using the demand and supply curves. The demand function (4.3) does not depend on variables in the SC repo market. The supply function (4.4) depends on the SC repo rate r_s. An increase (decrease) in r_s increases (decreases) the supply of the claim and shifts the supply curve right (left). With the unchanged demand curve, the shift of the supply curve to the right decreases the equilibrium price and increases the equilibrium volume. As mentioned in Section 1, the interest rate difference $r - r_s$ corresponds to funding the asset cost for the liability side (i.e. the option seller). An increase in r_s reduces the funding cost, making it easier for a seller enter into a derivatives contract and, thus, leads to an increase in supply.

5 Equilibrium under Cash Collateralization

In order to demonstrate the effect of non-cash collateralization on the OTC derivatives market, we first derive equilibrium formulae for the market with cash collateralization. We follow the settings and notation introduced in Section 2, excluding that of cash collateralization. The derivation of the equilibria follows the procedure used in the previous section.

5.1 Optimization Problems

We assume there are two types of market participants, namely, long- and short-holders of the derivative. We again denote the long-holder by $j \in \mathcal{M}_l$, and the short-holder by $i \in \mathcal{M}_s$. Since we consider the case of cash collateralization, we omit the repo market.

We suppose that long-holders only have positive exposures in the derivatives contracts, and that short-holders post cash collateral to long-holders. We further assume that the posted cash amount

$C(0)$ is deposited into the risk-free asset, with constant interest rate r. Then, the amount deposited in the risk-free asset by long-holder j ($j \in \mathcal{M}_l$) is

$$M_0^j = x_j - \pi_j - k_j p + k_j C(0).$$

Then, the terminal wealth for the long-holder is given by

$$\begin{aligned} X_T^j =& M_0^j e^{rT} + \frac{\pi_j}{S_0} S_T + k_j H(T)(1 - 1_{D_i}) - k_j e^{rT} C(0)(1 - 1_{D_i}) \\ =& (x_j - \pi_j - k_j p)e^{rT} + \frac{\pi_j}{S_0} S_T + k_j g^c(T), \end{aligned} \tag{5.1}$$

for $i \in \mathcal{M}_s$, where

$$g^c(T) = H(T)(1 - 1_{D_i}) + e^{rT} C(0) 1_{D_i}.$$

The short-holder posts cash collateral owing to her/his default risk. Then, the amount deposited in the risk-free asset by short-holder i ($i \in \mathcal{M}_s$) is

$$M_0^i = x_i - \pi_i + k_i p - k_i C(0). \tag{5.2}$$

Then, the terminal wealth is given by

$$\begin{aligned} X_T^i =& M_0^i e^{rT} + \frac{\pi_i}{S_0} S_T - k_i H(T)(1 - 1_{D_i}) + k_i e^{rT} C(0)(1 - 1_{D_i}) \\ =& (x_i - \pi_i + k_i p)e^{rT} + \frac{\pi_i}{S_0} S_T - k_i g^c(T). \end{aligned} \tag{5.3}$$

In the case of cash collateralization, the agents determine the optimal volume of the derivatives to maximize their expected utilities. Then, the objective is

$$\max_{k_h} E[U_h(X_T^h)],$$

for h ($h \in \mathcal{M}_l \cup \mathcal{M}_s$).

5.2 Equilibria

As demonstrated in the previous sections, solving the optimization problems yields the demand and supply functions for the claim, and these lead to an equilibrium price and volume.

5.2.1 Demand Function

The demand function for the claim is derived by solving the long-holder's utility maximization problem:

$$\max_{k_l} E[U_l(X_T^l)].$$

For the expected utility (4.1), the FOC yields

$$-pe^{rT} + E[g^c(T)] - \gamma_l k_l Var[g^c(T)] - \gamma_l \frac{\pi_l}{S_0} Cov[S_T, g^c(T)] = 0.$$

Hence, we have the demand function,

$$k_l = \frac{1}{\gamma_l Var[g^c(T)]} \left(-pe^{rT} + E[g^c(T)] - \gamma_l \frac{\pi_l}{S_0} Cov[S_T, g^c(T)] \right). \tag{5.4}$$

5.2.2 Supply Function

The supply function for the claim is derived by solving the short-holder's utility maximization problem:

$$\max_{k_s} E[U_s(X_T^s)].$$

For the expected utility (4.2), the FOC yields

$$pe^{rT} - E[g^c(T)] - \gamma_s k_s Var[g^c(T)] + \gamma_s \frac{\pi_s}{S_0} Cov[S_T, g^c(T)] = 0.$$

Hence, we have the supply function,

$$k_s = \frac{1}{\gamma_s Var[g^c(T)]} \left(pe^{rT} - E[g^c(T)] + \gamma_s \frac{\pi_s}{S_0} Cov[S_T, g^c(T)] \right). \tag{5.5}$$

5.2.3 Equilibria

From the equilibrium condition

$$k_l = k_s,$$

we derive the equilibrium price and volume for the derivatives contract. Substituting (5.4) and (5.5) into the above equilibrium condition, we have equilibrium price p^* for the claim with cash collateralization,

$$p^* = \frac{1}{e^{rT}} E[g^c(T)] - \frac{\gamma}{e^{rT}} \frac{\pi_l + \pi_s}{S_0} Cov[S_T, g^c(T)]. \tag{5.6}$$

Next, substituting (5.6) into (5.4) or (5.5) gives the equilibrium volume k^* of the claim with cash collateralization,

$$k^* = \frac{\gamma_s \pi_s - \gamma_l \pi_l}{(\gamma_l + \gamma_s) S_0} \frac{Cov[S_T, g^c(T)]}{Var[g^c(T)]}. \tag{5.7}$$

Remark 5.1. *In the case of non-cash collateralization, let us suppose that collateral asset B is a zero-coupon discount bond, with maturity date T and $B_T = 1$. Then, the terminal value of B is not random, and the equilibrium price (4.5) and volume (4.6) are reduced to the price (5.6) and volume (5.7) respectively. That is, the equilibrium under non-cash collateralization reduces to the equilibrium with cash collateralization.*

6 Numerical Example

6.1 Continuous-Time Model

Here, we consider a filtered probability space $(\Omega, P, \mathcal{F}, \mathcal{F}_t)$. Furthermore, we introduce a four-dimensional standard Brownian motion, $W_t = (W_{1t}, \ldots, W_{4t})$. We suppose that filtration $\mathcal{F}_t^W$ is generated by the Brownian motion, as follows:

$$\mathcal{F}_t^W = \sigma(W_s; s \leq t).$$

We set a default event, following Schönbucher (2003). We denote the time of the short-holder's default by τ, and define it as the time when Poisson process N first jumps; that is,

$$\tau = \inf\{t > 0 | N_t > 0\}.$$

Then, the default indicator function 1_{D_s} of the short-holder is given by

$$1_{D_s} = 1_{\tau \le T}.$$

We also suppose that Poisson process N follows a Cox process, that is, its intensity process λ is driven stochastically. We denote the filtration generated by the Poisson process by $\mathcal{F}_t^N$; that is,

$$\mathcal{F}_t^N = \sigma(N_s; s \le t).$$

Then, filtration $\mathcal{F}_t$ is defined by

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^N.$$

Next, we assume that price processes in our financial market are driven by the following stochastic processes. The stock price process with stochastic volatility is

$$dS_t = S_t(\mu dt + \sigma_t dW_{1t}),$$

where $\sigma_t = \sqrt{Y_t}$, and Y_t is

$$dY_t = \kappa_Y(a_Y - Y_t)dt + b_Y\sqrt{Y_t}(\rho_Y dW_{1t} + \sqrt{1-\rho_Y^2}dW_{2t}). \tag{6.1}$$

This stochastic volatility model is known as the Heston model (Heston 1993). We use this model as an example. The price process of the collateral asset is given by

$$dB_t = B_t\{\mu_B dt + \sigma_B(\rho_B dW_{1t} + \sqrt{1-\rho_B^2}dW_{3t})\}. \tag{6.2}$$

The stochastic intensity process is driven by the so-called CIR-type process (Cox et al. 1985),

$$d\lambda_t = \kappa_\lambda(a_\lambda - \lambda_t)dt + b_\lambda\sqrt{\lambda_t}(\rho_\lambda dW_{1t} + \sqrt{1-\rho_\lambda^2}dW_{4t}). \tag{6.3}$$

Remark 6.1 (Right/Wrong Way Risk)**.** *Parameter ρ_λ indicates the correlation between the underlying asset price and the default intensity. Since we consider a call option, when ρ_λ is positive, the increase in the option value tends to increase the default intensity, and vice versa. Therefore, a positive ρ_λ implies a wrong-way risk, and a negative value corresponds to a right-way risk.*

We use the Euler discretization to simulate the above processes. For (6.1) and (6.3), we utilize the so-called Milstein discretization (c.f. Chapter 2 in Gatheral 2006) to avoid negative volatility and intensity. Using the Milstein method, we discretize processes (6.1) and (6.3) as follows:

$$\begin{aligned} Y_{t+\Delta t} =& Y_t + \kappa_Y(a_Y - Y_t)\Delta t + b_Y\sqrt{Y_t}(\rho_Y\sqrt{\Delta t}G_1 + \sqrt{1-\rho_Y^2}\sqrt{\Delta t}G_2) \\ &+ \frac{b_Y^2\Delta t}{4}(\rho_Y^2 G_1^2 + 2\rho_Y\sqrt{1-\rho_Y^2}G_1G_2 + (1-\rho_Y^2)G_2^2 - 1), \end{aligned}$$

with $4\kappa_Y a_Y/b_Y^2 > 1$, and

$$\begin{aligned} \lambda_{t+\Delta t} =& \lambda_t + \kappa_\lambda(a_\lambda - \lambda_t)\Delta t + b_\lambda\sqrt{\lambda_t}(\rho_\lambda\sqrt{\Delta t}G_1 + \sqrt{1-\rho_\lambda^2}\sqrt{\Delta t}G_3) \\ &+ \frac{b_\lambda^2\Delta t}{4}(\rho_\lambda^2 G_1^2 + 2\rho_\lambda\sqrt{1-\rho_\lambda^2}G_1G_3 + (1-\rho_\lambda^2)G_3^2 - 1), \end{aligned}$$

with $4\kappa_\lambda a_\lambda/b_\lambda^2 > 1$, where G_n $(n = 1, 2, 3)$ are i.i.d. $\mathcal{N}(0, 1)$.

	Non-Cash			Cash		
ϕ	Price	Volume	Size	Price	Volume	Size
0%	3.02	145.21	438.43	3.02	145.21	438.43
10%	3.02	146.66	443.33	3.02	147.72	446.38
20%	3.03	147.80	447.49	3.03	150.05	453.95
30%	3.03	148.60	450.74	3.03	152.14	460.92
40%	3.04	149.12	452.91	3.03	154.05	467.09
50%	3.04	149.36	454.04	3.03	155.71	472.32
60%	3.05	149.21	454.55	3.04	157.09	477.31
70%	3.05	148.84	454.04	3.04	158.24	481.22
80%	3.06	148.18	452.76	3.04	159.09	484.35
90%	3.06	147.23	450.68	3.05	159.64	486.69
100%	3.06	146.15	447.72	3.05	159.91	487.66
110%	3.07	144.73	443.91	3.05	159.86	487.92
120%	3.07	143.10	439.84	3.06	159.52	487.69
130%	3.08	141.28	434.77	3.06	158.87	486.13
140%	3.08	139.38	429.65	3.06	157.96	483.94
150%	3.08	137.30	423.32	3.06	156.72	480.12
160%	3.09	134.94	417.18	3.07	155.25	476.73
170%	3.09	132.65	410.55	3.07	153.52	471.81
180%	3.10	130.24	403.59	3.08	151.54	466.19
190%	3.10	127.74	396.01	3.08	149.33	459.48
200%	3.11	125.17	388.77	3.08	146.99	453.03

Table 1: The equilibrium price, volume, and market size under non-cash collateralization and cash collateralization at $\rho_\lambda = -0.5$ (i.e. right-way risk case). The column 'Non-Cash' corresponds to the case of non-cash collateralization, and the column 'Cash' relates to the case of cash collateralization. Then, 'Size' shows the market size of our OTC option market, defined as the product of price and volume.

6.2 Numerical Result

We implement our equilibrium formulae (4.5), (4.6), (5.6), and (5.7) in a Monte-Carlo simulation using the above asset price processes in above. Each term in the equilibrium formulae, such as expectation, variance, and so on, in the continuous time model in Section 6.1, is presented in the Appendix.

In particular, we observe the effects of non-cash collateralization on the equilibrium by changing some of the parameters. The unchanged parameters used in the simulation are as follows: $\mu = 0.1$, $S_0 = 10.0$, $Y_0 = 0.04$, $\kappa_Y = 1.0$, $a_Y = 0.04$, $b_Y = 0.2$, $\rho_Y = -0.5$, $\mu_B = 0.05$, $\sigma_B = 0.1$, $B_0 = 95.0$, $\rho_B = 0.75$, $\lambda_0 = 0.04$, $\kappa_\lambda = 1.0$, $a_\lambda = 0.05$, $b_\lambda = 0.2$, $\pi_l = 10.0$, $\pi_s = 2000.0$, $r = 0.05$, $r_s = 0.03$, $T = 0.25$ and $K = 7.0$. In addition, the frequency of Monte-Carlo simulations are run 1,000,000 times, and we divide one year into 500 time grids.

6.2.1 Comparison with Cash Collateralization

We first compare the results for the cases of non-cash collateralization and cash collateralization. Table 1 shows the equilibrium prices, volumes, and market sizes for each of the cases and for each

	Non-Cash			Cash		
ϕ	Price	Volume	Size	Price	Volume	Size
0%	3.02	143.75	433.84	3.02	143.75	433.84
10%	3.02	145.39	439.28	3.02	146.39	442.11
20%	3.03	146.71	443.94	3.02	148.83	449.99
30%	3.03	147.69	447.68	3.03	151.05	457.31
40%	3.04	148.37	450.32	3.03	153.10	463.87
50%	3.04	148.77	451.90	3.03	154.89	469.50
60%	3.04	148.75	452.79	3.04	156.41	474.88
70%	3.05	148.50	452.65	3.04	157.71	479.19
80%	3.05	147.96	451.70	3.04	158.71	482.75
90%	3.06	147.11	449.91	3.05	159.41	485.52
100%	3.06	146.12	447.22	3.05	159.83	486.92
110%	3.06	144.77	443.63	3.05	159.94	487.63
120%	3.07	143.21	439.75	3.05	159.74	487.83
130%	3.07	141.44	434.84	3.06	159.25	486.71
140%	3.08	139.59	429.86	3.06	158.48	484.95
150%	3.08	137.54	423.65	3.06	157.39	481.56
160%	3.09	135.21	417.59	3.07	156.06	478.58
170%	3.09	132.95	411.03	3.07	154.47	474.08
180%	3.10	130.55	404.12	3.07	152.61	468.84
190%	3.10	128.07	396.60	3.07	150.54	462.53
200%	3.10	125.51	389.38	3.08	148.31	456.42

Table 2: The equilibrium price, volume, and market size under non-cash collateralization and cash collateralization at $\rho_\lambda = 0.9$ (i.e. wrong-way risk case). The column 'Non-Cash' corresponds to the case of non-cash collateralization, and the column 'Cash' relates to the case of cash collateralization. Then, 'Size' shows the market size of our OTC option market, defined as the product of price and volume.

coverage ratio ϕ, when $\rho_\lambda = -0.5$ (i.e. the right-way risk case). The market size is defined as the product of price and volume. For both cases, the option price monotonically increases when the coverage ratio increases. On the other hand, the relation between the coverage ratio and the volume (and, thus, the coverage ratio and the size), is not uniform. When the coverage ratio increases, the volume monotonically increases in ϕ by a relatively small amount, and then monotonically decreases. For the case of cash collateralization, Takino (2016b) obtains the same result. The table also shows that the price under non-cash collateralization is larger than or equal to the price under cash collateralization, for all ϕ. The volume and market size under non-cash collateralization are smaller than those under cash collateralization, except in the case of $\phi = 0\%$. Since no collateral is posted (or received) at $\phi = 0\%$, the values of price and volume are same between the non-cash and cash collateralization case.

Table 2 lists the equilibrium prices, volumes, and market sizes for both cases for $\rho_\lambda = 0.9$ (i.e. the wrong-way case). We have almost the same characteristics as those of the right-way risk case. From the table, the price under non-cash collateralization is larger than or equal to the price under cash collateralization, for all ϕ. The volume and market size under non-cash collateralization are smaller than those under cash collateralization, except in the case of $\phi = 0\%$.

Volume (k)	$\sigma_B = 0.1$ Demand	$\sigma_B = 0.1$ Supply	$\sigma_B = 0.5$ Demand	$\sigma_B = 0.5$ Supply	Cash Collateralization Demand	Cash Collateralization Supply
0	3.23	2.76	3.23	2.48	3.23	2.83
10	3.22	2.78	3.22	2.56	3.22	2.85
20	3.21	2.81	3.21	2.64	3.21	2.87
30	3.20	2.84	3.20	2.73	3.20	2.89
40	3.19	2.87	3.19	2.81	3.19	2.91
50	3.17	2.90	3.17	2.89	3.18	2.93
60	3.17	2.93	3.16	2.98	3.17	2.95
70	3.16	2.96	3.15	3.06	3.16	2.97
80	3.15	2.98	3.14	3.14	3.15	2.99
90	3.14	3.01	3.13	3.23	3.14	3.01
100	3.12	3.04	3.12	3.31	3.12	3.03
110	3.11	3.07	3.11	3.39	3.11	3.05
120	3.10	3.10	3.10	3.48	3.10	3.07
130	3.09	3.13	3.09	3.56	3.09	3.09
140	3.08	3.16	3.08	3.64	3.08	3.11
150	3.07	3.18	3.07	3.72	3.07	3.13
160	3.06	3.21	3.06	3.81	3.06	3.16
170	3.05	3.24	3.05	3.89	3.05	3.18
180	3.04	3.27	3.04	3.97	3.04	3.20
190	3.03	3.29	3.03	4.05	3.03	3.21
200	3.02	3.33	3.02	4.14	3.02	3.24

Table 3: The values of the demand and supply functions. The columns labeled '$\sigma_B = 0.1$' and '$\sigma_B = 0.5$' relates to the case of non-cash collateralization.

6.2.2 Demand and Supply Functions

Next, we next observe the effect of non-cash collateralization on demand and supply by comparing it to the case of cash collateralization. Table 3 shows the reservation prices for participants as a function of volume k for the two cases, that is, the table represents the demand and supply functions numerically. We set $\sigma_B = 0.1, 0.5$ in (6.2) and the coverage ratio $\phi = 100\%$. For the case of $\sigma_B = 0.5$, we consider a situation where the collateral asset price is more volatile. The columns of '$\sigma_B = 0.1$' and '$\sigma_B = 0.5$' in Table 3 correspond to the non-cash collateralization case.

From the table, we find no significant difference in the reservation price of the demand side between two cases. That is, there is no effect of non-cash collateralization on demand in our model. On the other hand, we observe a difference in the reservation price of the supply side between two cases. To observe this, we plot the supply function as a function of volume k in Figure 1. From the figure, the change in the slope of the supply curve is more significant than the shift of the supply curve. In fact, from (4.4), the supply function under non-cash collateralization is

$$\begin{aligned} p =& \frac{1}{e^{rT}} \left\{ \gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T \right] k + E[g(T)] - \frac{C(0)}{B_0}(e^{r_s T} B_0 - E[B_T]) \right. \\ & \left. - \gamma_s \frac{\pi_s}{S_0} Cov\left[S_T, g(T) + \frac{C(0)}{B_0} B_T \right] \right\}, \end{aligned}$$

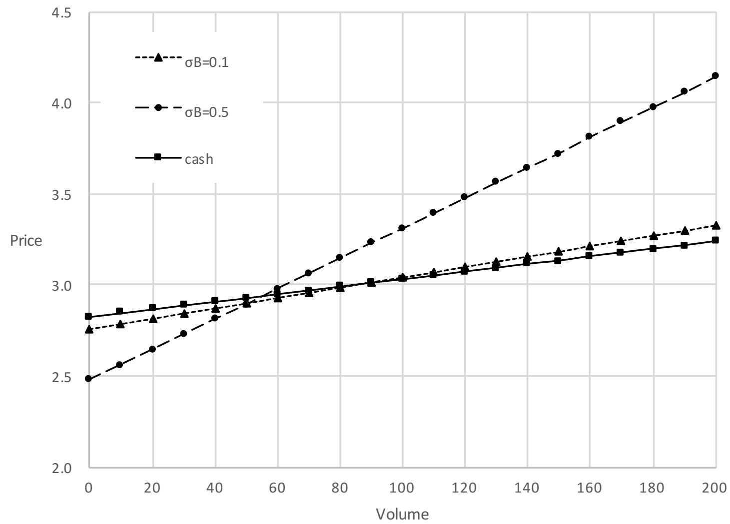

Figure 1: Supply functions for non-cash collateralization and cash collateralization. Dashed lines correspond to the case of non-cash collateralization (labeled 'σ_B'), and the solid line relates to the cash collateralization case (labeled 'Cash'). The data are based on Table 3.

and the supply function in the cash collateralization case is

$$p = \frac{1}{e^{rT}} \left\{ \gamma_s Var\left[g^c(T)\right] k + E[g^c(T)] - \gamma_s \frac{\pi_s}{S_0} Cov\left[S_T, g^c(T)\right] \right\},$$

from (5.5). The gradient of the supply function with non-cash collateralization case is

$$\frac{1}{e^{rT}} \gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right], \tag{6.4}$$

and the gradient of the supply function under cash collateralization is

$$\frac{1}{e^{rT}} \gamma_s Var\left[g^c(T)\right].$$

It is easy to deduce that

$$\frac{1}{e^{rT}} \gamma_s Var\left[g(T) + \frac{C(0)}{B_0} B_T\right] > \frac{1}{e^{rT}} \gamma_s Var\left[g^c(T)\right],$$

when B_T is random and the covariance of $g(T)$ and B_T is positive. Note that, the positive covariance between $g(T)$ and B_T is satisfied because the option is a call type and the correlation between S_t and B_t is positive[2] for $0 \leq t \leq T$.

In Section 6.2.1, we find that the OTC option market contracts when the collateral asset is changed from cash to the risky asset. The figure implies that this change arises from the change in

[2]In the numerical simulation, we set $\rho_B = 0.75$.

ϕ	Proposed Model	Lou	Difference
0%	3.019	3.046	0.89%
10%	3.023	3.049	0.86%
20%	3.028	3.052	0.81%
30%	3.033	3.056	0.75%
40%	3.037	3.059	0.71%
50%	3.040	3.061	0.69%
60%	3.046	3.065	0.62%
70%	3.051	3.068	0.57%
80%	3.055	3.071	0.52%
90%	3.061	3.075	0.46%
100%	3.063	3.077	0.45%
110%	3.067	3.080	0.41%
120%	3.074	3.084	0.34%
130%	3.077	3.087	0.31%
140%	3.083	3.090	0.26%
150%	3.083	3.092	0.27%
160%	3.092	3.097	0.18%
170%	3.095	3.100	0.16%
180%	3.099	3.103	0.14%
190%	3.100	3.105	0.15%
200%	3.106	3.109	0.10%

Table 4: The equilibrium price from the proposed formula (4.5) and that of Lou (2017) (6.5) at $\rho_\lambda = -0.5$. The column of 'Proposed Model' corresponds to the pricing formula (4.5), and the column of 'Lou' relates to pricing formula (6.5). 'Difference' refers to the price difference between the two formulae, calculated as $\frac{\text{Lou (2017)}-\text{Our Model}}{\text{Our Model}} \times 100\%$.

the slope of the supply curve, and not from the change in the demand curve. Furthermore, from (6.4), the gradient becomes steeper when B_T becomes more volatile if the covariance of $g(T)$ and B_T is positive (i.e. the line of '$\sigma_B = 0.5$' is steeper than the line of '$\sigma_B = 0.1$').

6.2.3 Comparison with Lou's Model

We compare our pricing formula to the pricing model of Lou (2017) for the non-cash collateralization case. Lou (2017) considers risk-neutral pricing of an option when the liability side sources the asset posted to the asset side as collateral from the SC repo market, and derives a BS-PDE incorporating the sourced the asset. Lou's model shows how to select the discount rate in the BS formula when cash is not used as collateral. Furthermore, Lou (2017) demonstrates that the discount rate is given by the weighted average of the funding liquidity rate (i.e. interest rate) of the agent, the haircut, and the repo rate.

Applying the pricing concept of Lou (2017) to our model, we derive the option pricing formula with non-cash collateralization as

$$\text{Option Price} = E[\mathcal{E}^s(T)g(T)]. \tag{6.5}$$

Recall that the proposed model does not require collateral in addition to the posted collateral asset, even if the contract is not fully collateralized. Therefore, the funding cost of the agent with a negative exposure is only the SC repo rate r_s, and pricing kernel $\mathcal{E}^s$ is used in formula (6.5) instead of $\mathcal{E}$.

Table 4 shows the option prices from our equilibrium pricing formula (4.5) and that from (6.5) at $\rho_\lambda = -0.5$. The table shows there are few price differences. At $\phi = 0\%$, the price of our model is higher than the price of Lou (2017) by 0.89% which difference is the largest compared with another differences. At $\phi = 200\%$, the price of our model is the closest to the price of Lou (2017). Therefore, the price differences between the proposed equilibrium formula and the risk-neutral method of Lou (2017) are not large.

7 Concluding Remarks

In this study, we proposed an equilibrium pricing model for an OTC option with non-cash collateralization. We analysed how the OTC derivatives market is constructed when non-cash assets are used as collateral. The SC repo rate corresponds the cost of funding the asset, and a higher repo rate (lower funding cost) increases the supply of the liability side, and decreases the price and increases the volume on the claim. We compared the price of our model to an existing pricing approach, finding that our equilibrium price is close to that of the existing risk-neutral approach. Finally, we compared the market sizes for the cases of non-cash collateralization and cash collateralization. The result show that the market size under non-cash collateralization is smaller than that under cash collateralization.

On the other hand, it has been reported that it is the market with a wide asset class of collateral that grows rather than a market with restricted asset class of collateral. Our result somewhat contradicts this finding. Therefore, as future work, we would like to consider a framework in which the agent who posts collateral is able to select cash collateral or non-cash collateral.

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A Building Blocks

In this appendix, we provide explicit formulations for the terms of expectation, variance, and covariance in the equilibrium formulae (4.5), (4.6), (5.6), and (5.7) under the continuous-time model defined in Section 6.1.

A.1 Non-Cash Collateralization Case

First, the expectation of g is

$$\begin{aligned}
E[g(T)] =& E\left[H(T)(1-1_{\tau\leq T})+\frac{C(0)}{B_0}B_T 1_{\tau\leq T}\right]\\
=& E[H(T)1_{\tau>T}]+\frac{C(0)}{B_0}(E[B_T]-E[B_T 1_{\tau>T}])\\
=& E[E[H(T)1_{\tau>T}|\mathcal{F}^W]]+\frac{C(0)}{B_0}(E[B_T]-E[E[B_T 1_{\tau>T}|\mathcal{F}^W]])\\
=& E[H(T)E[1_{\tau>T}|\mathcal{F}^W]]+\frac{C(0)}{B_0}(E[B_T]-E[B_T E[1_{\tau>T}|\mathcal{F}^W]])\\
=& E[e^{-\int_0^T\lambda_t dt}H(T)]+\frac{C(0)}{B_0}(E[B_T]-E[e^{-\int_0^T\lambda_t dt}B_T]).
\end{aligned}$$

Next, the variance is

$$Var\left[g(T)+\frac{C(0)}{B_0}B_T\right] = Var[g(T)]+\left(\frac{C(0)}{B_0}\right)^2 Var[B_T]+2\frac{C(0)}{B_0}Cov[g(T),B_T],$$

where

$$\begin{aligned}
Var[g(T)] =& Var\left[H(T)1_{\tau>T}+\frac{C(0)}{B_0}B_T(1-1_{\tau>T})\right]\\
=& Var[H(T)1_{\tau>T}]+\left(\frac{C(0)}{B_0}\right)^2 Var[B_T(1-1_{\tau>T})]\\
&+2\frac{C(0)}{B_0}Cov[H(T)1_{\tau>T},B_T(1-1_{\tau>T})],
\end{aligned}$$

with

$$\begin{aligned}
Var[H(T)1_{\tau>T}] =& E[(H(T)1_{\tau>T})^2]-E[H(T)1_{\tau>T}]^2\\
=& E[H^2(T)1_{\tau>T}]-E[1_{\tau>T}]^2\\
=& E[e^{-\int_0^T\lambda_t dt}H^2(T)]-E[e^{-\int_0^T\lambda_t dt}H(T)]^2,
\end{aligned}$$

$$\begin{aligned}
Var[B_T(1-1_{\tau>T})] =& Var[B_T]+Var[B_T 1_{\tau>T}]-2Cov[B_T,B_T 1_{\tau>T}]\\
=& E[B_T^2]-E[B_T]^2+E[(B_T 1_{\tau>T})^2]-E[B_T 1_{\tau>T}]^2\\
&-2(E[B_T^2 1_{\tau>T}]-E[B_T]E[B_T 1_{\tau>T}])\\
=& E[B_T^2]-E[B_T]^2+E[e^{-\int_0^T\lambda_t dt}B_T^2]-E[e^{-\int_0^T\lambda_t dt}B_T]^2
\end{aligned}$$

and

$$\begin{aligned}Cov[H(T)1_{\tau>T}, B_T(1-1_{\tau>T})] =& Cov[H(T)1_{\tau>T}, B_T] - Cov[H(T)1_{\tau>T}, B_T 1_{\tau>T}]\\ =& E[H(T)1_{\tau>T}B_T] - E[H(T)1_{\tau>T}]E[B_T]\\ &- (E[H(T)B_T(1_{\tau>T})^2] - E[H(T)1_{\tau>T}]E[B_T 1_{\tau>T}])\\ =& E[e^{-\int_0^T \lambda_t dt}H(T)B_T] - E[e^{-\int_0^T \lambda_t dt}H(T)]E[B_T]\\ &- (E[e^{-\int_0^T \lambda_t dt}H(T)B_T] - E[e^{-\int_0^T \lambda_t dt}H(T)]E[e^{-\int_0^T \lambda_t dt}B_T]).\end{aligned}$$

Then, the variance of $g(T) + \frac{C(0)}{B_0}B_T$ is calculated as

$$Var\left[g(T) + \frac{C(0)}{B_0}B_T\right] = Var[g(T)] + \left(\frac{C(0)}{B_0}\right)^2 Var[B_T] + 2\frac{C(0)}{B_0}Cov[g(T), B_T],$$

where

$$\begin{aligned}Cov[g(T), B_T] =& Cov\left[H(T)1_{\tau>T} + \frac{C(0)}{B_0}B_T(1-1_{\tau>T}), B_T\right]\\ =& Cov[H(T)1_{\tau>T}, B_T] + \frac{C(0)}{B_0}Cov[B_T(1-1_{\tau>T}), B_T],\end{aligned}$$

with

$$\begin{aligned}Cov[B_T(1-1_{\tau>T}), B_T] =& Cov[B_T, B_T] - Cov[B_T 1_{\tau>T}, B_T]\\ =& Var[B_T] - (E[B_T^2 1_{\tau>T}] - E[B_T 1_{\tau>T}]E[B_T]).\end{aligned}$$

Finally, we give the covariances as

$$\begin{aligned}Cov[S_T, g(T)] =& Cov\left[S_T, H(T)1_{\tau>T} + \frac{C(0)}{B_0}B_T(1-1_{\tau>T})\right]\\ =& Cov[S_T, H(T)1_{\tau>T}] + \frac{C(0)}{B_0}Cov[S_T, B_T(1-1_{\tau>T})]\\ =& E[S_T H(T)1_{\tau>T}] - E[S_T]E[H(T)1_{\tau>T}] + \frac{C(0)}{B_0}Cov[S_T, B_T(1-1_{\tau>T})],\end{aligned}$$

where

$$\begin{aligned}Cov[S_T, B_T(1-1_{\tau>T})] =& Cov[S_T, B_T] - Cov[S_T, B_T 1_{\tau>T}]\\ =& E[S_T B_T] - E[S_T]E[B_T] - (E[S_T B_T 1_{\tau>T}] - E[S_T]E[B_T 1_{\tau>T}])\\ =& E[S_T B_T] - E[S_T]E[B_T]\\ &- (E[e^{-\int_0^T \lambda_t dt}S_T B_T] - E[S_T]E[e^{-\int_0^T \lambda_t dt}B_T]),\end{aligned}$$

and

$$Cov\left[S_T, g(T) + \frac{C(0)}{B_0}B_T\right] = Cov[S_T, g(T)] + \frac{C(0)}{B_0}Cov[S_T, B_T].$$

A.2 Cash Collateralization Case

We first calculate the expectation of $g^c(T)$.

$$\begin{aligned}E[g^c(T)] =&E[H(T)1_{\tau>T} + e^{rT}C(0)(1-1_{\tau>T})] \\ =&E[H(T)1_{\tau>T}] + e^{rT}C(0)E[1-1_{\tau>T}] \\ =&E[E[H(T)1_{\tau>T}|\mathcal{F}^W]] + e^{rT}C(0)E[1-E[1_{\tau>T}|\mathcal{F}^W]] \\ =&E[e^{-\int_0^T \lambda_t dt}H(T)] + e^{rT}C(0)(1-E[e^{-\int_0^T \lambda_t dt}]).\end{aligned}$$

Next, the variance of $g^c(T)$ is

$$\begin{aligned}Var[g^c(T)] =&Var[H(T)1_{\tau>T} + e^{rT}C(0)(1-1_{\tau>T})] \\ =&Var[H(T)1_{\tau>T}] + (e^{rT}C(0))^2 Var[1-1_{\tau>T}] + 2e^{rT}C(0)Cov[H(T)1_{\tau>T}, 1-1_{\tau>T}],\end{aligned}$$

where

$$\begin{aligned}Var[1-1_{\tau>T}] =&Var[1_{\tau>T}] \\ =&E[(1_{\tau>T})^2] - E[1_{\tau>T}]^2 \\ =&E[1_{\tau>T}] - E[1_{\tau>T}]^2,\end{aligned}$$

and

$$\begin{aligned}Cov[H(T)1_{\tau>T}, 1-1_{\tau>T}] =& - Cov[H(T)1_{\tau>T}, 1_{\tau>T}] \\ =& - (E[H(T)(1_{\tau>T})^2] - E[H(T)1_{\tau>T}]E[1_{\tau>T}]) \\ =& - (E[H(T)1_{\tau>T}] - E[H(T)1_{\tau>T}]E[1_{\tau>T}]).\end{aligned}$$

Note that $Var[H(T)1_{\tau>T}]$ is calculated as in Section A.1.