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Comparison of utility indifference pricing and mean-variance approach under a normal mixture distribution with time-varying volatility

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Abstract

We evaluate a utility indifference price with an exponential utility function, which we call a risk-sensitive value measure, under a normal mixture distribution with time-varying volatility. We compare the risk-sensitive value measure and mean-variance approach and provide an empirical application.

JEL codes; G11; G32; C13; C46; C58
Keywords; Value measure; Utility indifference pricing; Normal mixture; GARCH

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1 Introduction

How to evaluate uncertain projects, future cash flows, random returns, etc. is fundamental in finance. Recently Miyahara (2010) proposed utility indifference pricing with an exponential utility function to deal with this important problem and proved this utility indifference pricing satisfies several desirable properties an evaluation function of random variables such as uncertain projects ought to satisfy. Furthermore, it was shown (see Theorem 3.2.8 of Rolski et al. (1999) for its proof) that the exponential utility function $u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$ and the utility indifference price with the exponential utility function is the only utility function and the only utility indifference price among $\boldsymbol{C}^2$-class of utility functions under a certain condition. Therefore, the utility indifference price with the exponential utility function is the only possible candidate for the suitable value measure. Miyahara (2010) called the utility indifference price with the exponential utility function a *risk-sensitive value measure* (RSVM) because it gives values sensitively to the underlying risk of random variables such as uncertain projects, future cash flows, random returns, etc. In other words, the RSVM of a random variable becomes positive if it is attractive or profitable and negative otherwise.

Hodoshima et al. (2018) made a comparison of the RSVM and mean-variance (MV) approach under a normal mixture distribution to present numerical examples and an empirical example concerning the RSVM and MV approach. They showed the RSVM is more sensitive to the underlying risk of a financial asset in question than the MV approach. Normal mixture distributions are a practical and flexible class of distributions which can capture symmetric distributions as well as skewed, leptokurtic, and multimodal distributions often observed in financial instruments (cf., e.g., Everitt and Hand (1981), Titterington et al. (1985), McLachlan and Peel (2000), Kon (1984), Ritchey (1990), Chin et al. (1999), Brigo and Mercurio (2001), and Alexander (2004)). It is also true that any continuous distribution can be approximated by a finite discrete normal mixture distribution. On the other hand, it is a stylized fact that the volatility of asset returns is time varying and clustered over time. A normal mixture distribution is apparently a time-independent model which assumes that observations are independent over time. Therefore, a normal mixture distribution is not suitable to capture the feature of volatility clustering. In this paper, we extend the result of Hodoshima et al. (2018), i.e., compare the RSVM and MV-approach, when the underlying distribution of a financial asset is given by a discrete normal mixture distribution with time-varying volatility. By combining a discrete normal mixture distribution and time-varying volatility, we make the underlying distribution of a financial asset more realistic. We assume conditional variance to follow the generalized autoregressive conditional heteroskedastic (GARCH) model proposed by Bollerslev (1986), which is a generalized model of the autoregressive conditional heteroskedastic (ARCH) model proposed by a seminal paper Engle (1982).

There are several ways to incorporate the GARCH model into a discrete normal mixture distribution. In this paper, we adopt the specification of a discrete normal mixture distribution with the GARCH conditional variance proposed by Haas et al. (2004) and Alexander and Lazar (2006), which allows for interdependence between variance components in each component of a discrete normal mixture distribution. To estimate this model, we employ an empirical characteristic function approach whose inference is based on the characteristic function of an underlying random variable, which can deal with the issue of estimability of a discrete normal mixture distribution by the maximum likelihood estimation. Namely we employ the continuous empirical characteristic function (CECF) method of Xu and Wirjanto (2010) and Xu and Knight (2011) (see also Knight and Yu (2002)) which minimizes the distance between a theoretical characteristic function and an empirical characteristic function with a continuous weighting function. In particular, we apply the CECF method of Xu and Wirjanto (2010), which provides a practical means to estimate the discrete normal mixture distribution with the GARCH volatility. Then we derive the RSVM and MV approach based on the estimate of the discrete normal mixture distribution with the GARCH volatility. We present a comparison of the RSVM and MV approach over time when the underlying distribution of a financial asset is given by the discrete normal mixture distribution with the GARCH(1,1) volatility.

We organize the paper as follows. Section 2 presents formulas of the RSVM and MV approach when the underlying distribution of a financial asset is given by the discrete normal mixture distribution with the GARCH(1,1) volatility. Section 3 presents an empirical example of the RSVM and MV approach using daily return data of the Dow

Jones Industrial Average (DJIA). Section 4 provides concluding comments.

2 The RSVM and MV approach under a discrete normal mixture distribution with the GARCH volatility

In this section we present the RSVM and MV approach under a discrete normal mixture distribution with the GARCH volatility.

We follow Xu and Wirjanto (2010) to assume the daily return $\boldsymbol{X}_t$ of an asset is given by

$$\boldsymbol{X}_t = \epsilon_t \tag{1}$$

where ϵ_t follows a mixture of K normal distributions with the time-varying volatility process

$$\epsilon_t | I_{t-1} \sim \pi_k N(\mu_k, \sigma_{k,t}^2) \tag{2}$$

for $t = 1, \cdots, T$ and $k = 1, \cdots, K$, where I_{t-1} is the information set up to time $t-1$, $0 \leq \pi_k \leq 1$ and $\sum_{k=1}^K \pi_k = 1$. The conditional variance of the k-th component is assumed to be given by a GARCH(m,n) process

$$\sigma_{k,t}^2 = \lambda_k + \sum_{i=1}^n \alpha_{ki} \epsilon_{t-i}^2 + \sum_{j=1}^m \beta_{kj} \sigma_{k,t-j}^2 \tag{3}$$

where the component conditional variances depend on previous innovations ϵ_{t-i} for $i = 1, \cdots, n$ as well as their own previous conditional variances. We follow Xu and Wirjanto (2010) and Haas et al. (2004) to make the component conditional variances not dependent on the previous conditional variances of other components. Then the conditional mean, variance, skewness, and kurtosis of X_t given the information set up to time $t-1$ are given respectively by

$$\begin{aligned}
\mu &= \sum_{k=1}^{K} \pi_k \mu_k \\
\sigma_t^2 &= \sum_{k=1}^{K} \pi_k (\sigma_{k,t}^2 + \mu_k^2) - \mu^2 \\
\tau_t &= \frac{1}{\sigma_t^3} \sum_{k=1}^{K} \pi_k (\mu_k - \mu) \left[3\sigma_{k,t}^2 + (\mu_k - \mu)^2 \right] \\
\kappa_t &= \frac{1}{\sigma_t^4} \sum_{k=1}^{K} \pi_k \left[3\sigma_{k,t}^4 + 6(\mu_k - \mu)^2 \sigma_{k,t}^2 + (\mu_k - \mu)^4 \right].
\end{aligned} \tag{4}$$

The RSVM of $\boldsymbol{X}_t$ is given by

$$-\frac{1}{\alpha} \ln E[e^{-\alpha \boldsymbol{X}_t}] \tag{5}$$

where α denotes the degree of risk aversion in the exponential utility function $u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$ (see Miyahara (2010)). When $\boldsymbol{X}_t$ has a mixture of K normal distributions with the GARCH(m,n) volatility given above, then the RSVM is given by

$$-\frac{1}{\alpha} \ln[\sum_{k=1}^{K} \pi_k \exp(-\mu_k \alpha + \sigma_{k,t}^2 \alpha^2/2)] \tag{6}$$

where $\sigma_{k,t}^2$ is given by equation (3) (cf. equation (6) of Hodoshima et al. (2018)). On the other hand, the MV of $\boldsymbol{X}_t$ is given by

$$E[\boldsymbol{X}_t] - \frac{1}{2}\alpha V[\boldsymbol{X}_t].$$

When the underlying distribution of $\boldsymbol{X}_t$ has a mixture of K normal distributions with the GARCH(m,n) volatility, then the MV of $\boldsymbol{X}_t$ is given by

$$MV(\alpha) = \sum_{i=1}^{K} \pi_k \mu_k - \frac{1}{2}\alpha \left\{ \sum_{k=1}^{K} \pi_k (\sigma_{k,t}^2 + \mu_k^2) - (\sum_{k=1}^{K} \pi_k \mu_k)^2 \right\} \tag{7}$$

where $\sigma_{k,t}^2$ is given by equation (3) (cf. equation (7) of Hodoshima et al. (2018)).

3 An empirical example of the DJIA

In this section, we present an empirical example of the RSVM and MV for the DJIA. We use the daily return data of the DJIA from January 2, 2008 until April 28, 2017.

We employ the CECF method of Xu and Wirjanto (2010) to estimate the underlying process. We present the estimate of the discrete normal mixture distribution with the GARCH volatility. In this empirical example, we only estimate, as the volatility process, the GARCH(1,1) volatility model which has been the most popular specification in many previous empirical studies.

We provide summary statistics of the daily return data of the DJIA at Table 1. It is a positively skewed distribution with heavy-tailed kurtosis.

We assume the daily return X_t follows the model given by equations (1), (2), and (3). Our model incorporates the GARCH process into a K-component normal mixture model. Hence it can capture two stylized facts of financial return data, i.e., distributional properties of skewness and high kurtosis as well as volatility clustering.

To estimate our model we use an empirical characteristic function approach. It is well known that the likelihood-based method has problems of estimation concerning normal mixture distributions. In other words, the likelihood function is not always bounded over its parameter space in normal mixture distributions (see, e.g., Quandt (1988), Quandt and Ramsey (1978), and Schmidt (1982)). An alternative estimation method is an empirical characteristic function approach that has important advantages over the likelihood-based method, i.e., the characteristic function is always uniformly bounded over the model's parameter space. Tran (1994) starts a discrete empirical function approach by matching a theoretical characteristic function with its empirical counterpart, i.e., by minimizing the distance between the theoretical characteristic function and the empirical characteristic function over a discrete fixed-grid-point set, in normal mixture models. Xu and Knight (2011) uses a CECF approach in normal mixture models which matches the theoretical characteristic function with its empirical counterpart continuously with a continuous weighting function. Xu and Wirjanto (2010) applies the CECF approach to normal mixture models with the GARCH volatility process. We employ the method of Xu and Wirjanto (2010) to compute the RSVM and MV approach in the normal mixture model with the GARCH(1,1) volatility process.

We present a two-components normal mixture model with the GARCH(1,1) volatility process[1]. The estimation result of a two-components normal mixture model with the GARCH(1,1) process is given at Table 2[2]. The first component is a dominant state with positive mean while the second component is a negative shock with large negative mean. $\alpha_{k1}(k = 1, 2)$, a response to the shock in the previous period, has a larger estimate in the negative shock state than in the dominant state. On the other hand, $\beta_{k1}(k = 1, 2)$, a response to the previous conditional variance, is similar in the two states. Parameters are insignificant except for α_{11} and β_{11}.

To show how the conditional variance in each component evolves over the entire sample, we provide graphs of the conditional variance against time at Figures 1-2 respectively for the conditional variance of the first and second component. We calculate the conditional variance using the parameter estimate from equation (3)[3]. We can see volatility clustering in the two components. Over the entire sample, the conditional variance of the second component fluctuates more strongly than that of the first component. This is reasonable because the second component represents a negative shock, whereas the first component represents a dominant state of ordinary variation. In the period of the financial crisis of 2008-2009, the conditional variance of the two components becomes large, and the conditional variance of the second component is even larger than that of the first component. These correspond to the fact that, in the period of the financial crisis of 2008-2009, the volatility of the stock market was soaring and the negative shock to the market was tremendous.

Based on the estimate of the model, we then compute the RSVM and MV approach in the DJIA. First we present graphs of the two measures over the entire sample for selective degree of risk aversion $\alpha = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5$. Figures 3-8 show those graphs respectively for $\alpha = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5$. There is almost no difference between the two measures when the degree of risk aversion, i.e., α, is 0.01. As the degree of risk

[1]We have also estimated a three-components normal mixture model with the GARCH(1,1) volatility. However we find the estimation result is the one where probability of the third component is very small so that a three-components normal mixture model with the GARCH(1,1) process is virtually the same as a two-components normal mixture model with the GARCH(1,1) process. Thus we only provide a two-components normal mixture model with the GARCH(1,1) process.

[2]To estimate the GARCH process, we set an initial value of both square of the error term and conditional variance equal to unconditional variance of the return observations.

[3]We derive the conditional variance from equation (3) with both the initial value of square of the error term and conditional variance equal to the unconditional variance of the return observations.

aversion increases, we can see some differences of the two measures particularly when the stock market underperformed, notably in the financial crisis of 2008-2009. The RSVM decreases more than the MV in the period of the financial crisis when the degree of risk aversion is large. This indicates that the RSVM responds more sensitively to the underlying risk of the stock market than the MV. Although Hodoshima et al. (2018) acknowledged a risk-sensitive property of the RSVM compared to the MV approach in the DJIA, they only reported the two measures once which correspond to the whole sample using the normal mixture distribution with no volatility process. Thus evolution of the RSVM and MV approach over time is a new feature not observed in previous studies.

In order to see the difference of the two measures more clearly, we provide graphs of the two measures for $\alpha = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5$ in the financial crisis period from September 2, 2008 till December 31, 2008. Figures 9-14 are those graphs respectively for $\alpha = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5$. We can see more clearly the difference of the two measures as the degree of risk aversion increases.

Figures 3-14 suggest the two measures of the RSVM and MV take similar values unless the market underperforms significantly and the degree of risk aversion is large. The RSVM becomes smaller than the MV when the above two conditions occur at the same time. We attribute this phenomenon to the underlying volatility of the market being high since we let mean of each component in the model remain constant but allow volatility of each component to vary in each observation. When the degree of risk aversion is small, the effect of volatility on the RSVM becomes small as we can see from equation (6) in Section 2. Only when the degree of risk aversion is large, the effect of volatility on the RSVM materializes. In our example of the DJIA, this happens even when the probability of the negative shock state, i.e., the second component, is as small as 0.0474. We expect the effect of volatility on the RSVM increases as the probability of the negative shock state becomes higher. In that case, the difference of the two measures appear even when the degree of risk aversion is small, i.e., when underlying investors are willing to take more risk.

4 Concluding Comments

We have presented a comparison of the RSVM and MV approach when the underlying distribution of uncertain projects, future cash flows, random returns, etc. is given by a normal mixture distribution with the time varying GARCH volatility process. With this underlying distribution we can capture both distributional properties of skewness and heavy-tail kurtosis and volatility clustering which are stylized facts in data of financial products.

We have provided the formulas of the RSVM and MV approach when the underlying distribution of an asset return is a normal mixture distribution with the time varying GARCH volatility process. Our comparison of the RSVM and MV approach using an empirical example of the DJIA shows that the RSVM decreases more than the MV approach when the market performs quite poorly as in the financial crisis of 2008-2009 if the degree of risk aversion is large. Thus the RSVM is more sensitive to the underlying risk of the financial target in question than the MV approach. This is a further confirmation of the risk-sensitive property of the RSVM already seen in Miyahara (2010) and Hodoshima et al. (2018) in the case of a normal mixture distribution with the time varying GARCH volatility process which can capture well stylized facts in financial data.

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Table 1: Summary Statistics of the Daily Return Data of the DJIA

name	mean	s.d.	skewness	kurtosis
DJIA	0.027	1.221	0.157	13.836

s.d. stands for standard deviation.

Table 2: Estimates of a Two-Component Normal Mixture Distribution with the GARCH(1,1) parameter for the daily return data of the DJIA

μ_1	μ_2	λ_1	α_{11}	β_{11}	λ_2	α_{21}	β_{21}	π_2
0.0583	-0.9845	0.0058	0.0880	0.8865	0.0000	0.3705	0.9056	0.0474
(0.0636)	(45.5908)	(0.1318)	(0.0063)	(0.0882)	(1.3073)	(11.2680)	(2.1144)	(0.3244)

Numbers shown inside brackets are asymptotoic standard errors.

Figure 1: Figure of the Realization of the Conditional Variance Process of the First Component over the Entire Sample

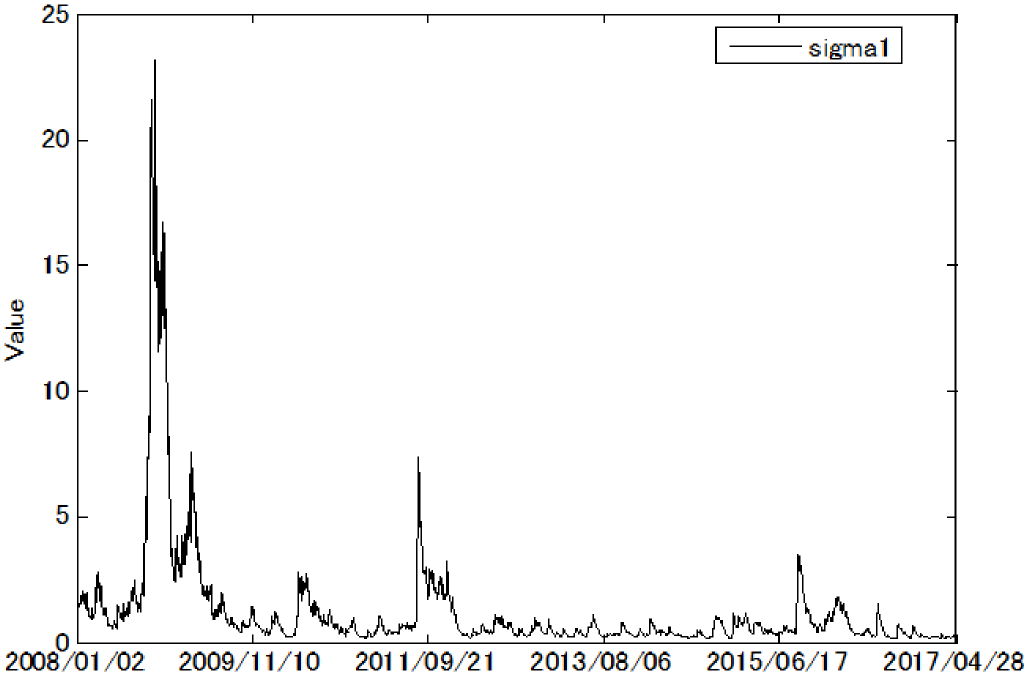


Figure 2: Figure of the Realization of the Conditional Variance Process of the Second Component over the Entire Sample

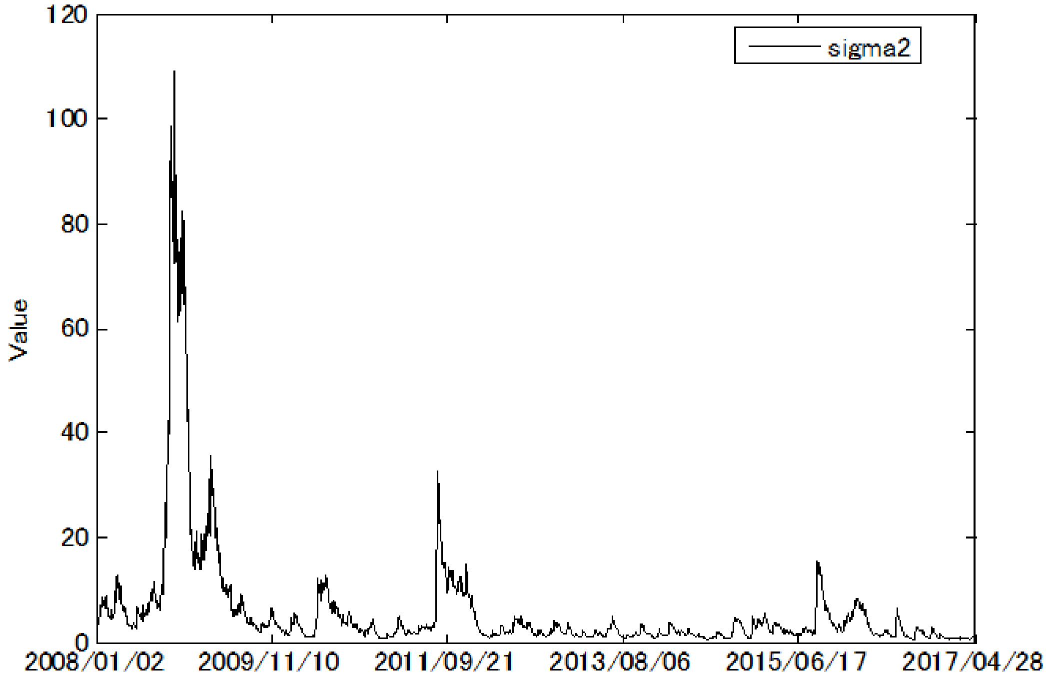


Figure 3: Figure of the RSVM and MV approach over the enire sample for $\alpha = 0.01$

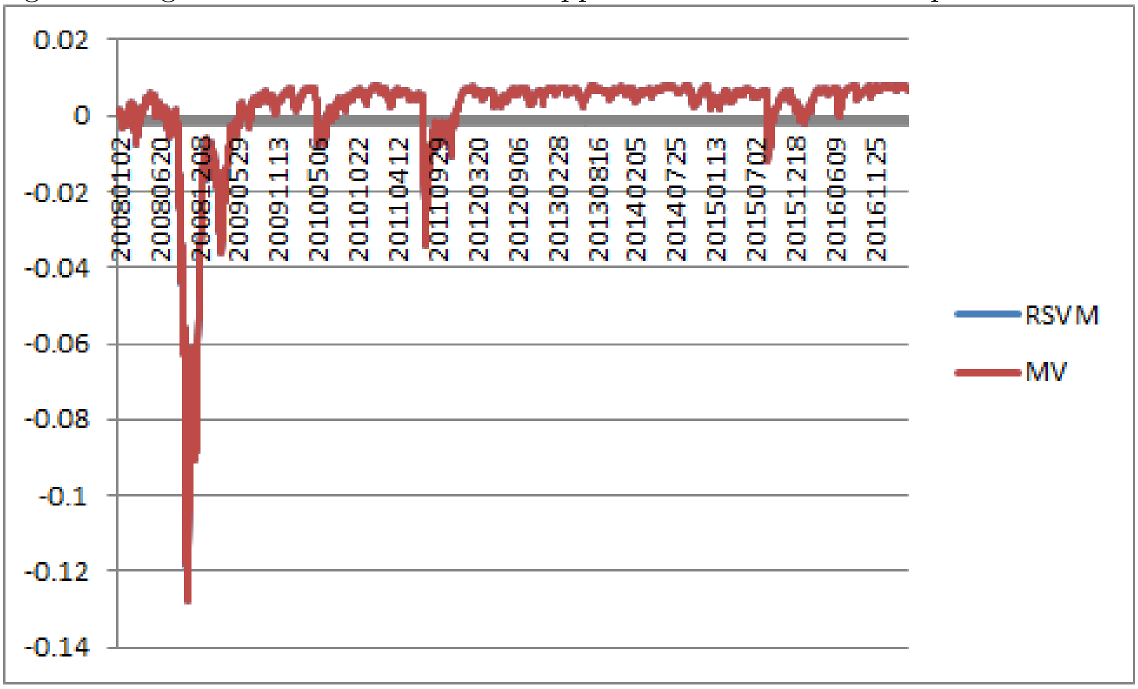


Figure 4: Figure of the RSVM and MV approach over the enire sample for $\alpha = 0.1$

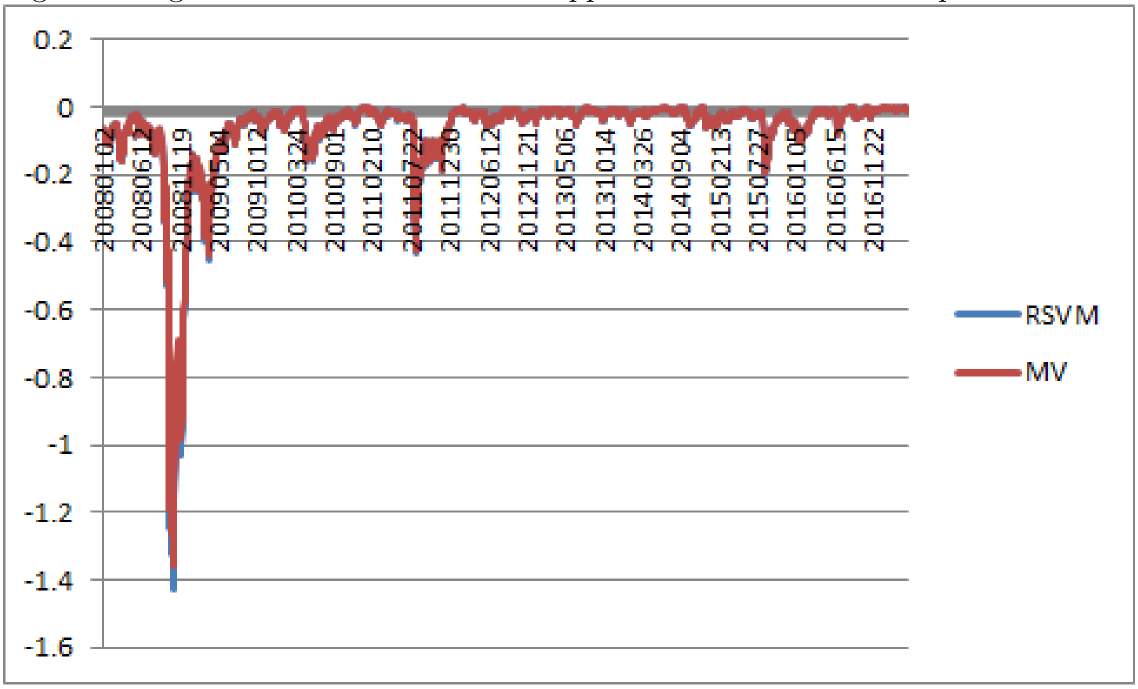


Figure 5: Figure of the RSVM and MV approach over the enire sample for $\alpha = 0.2$

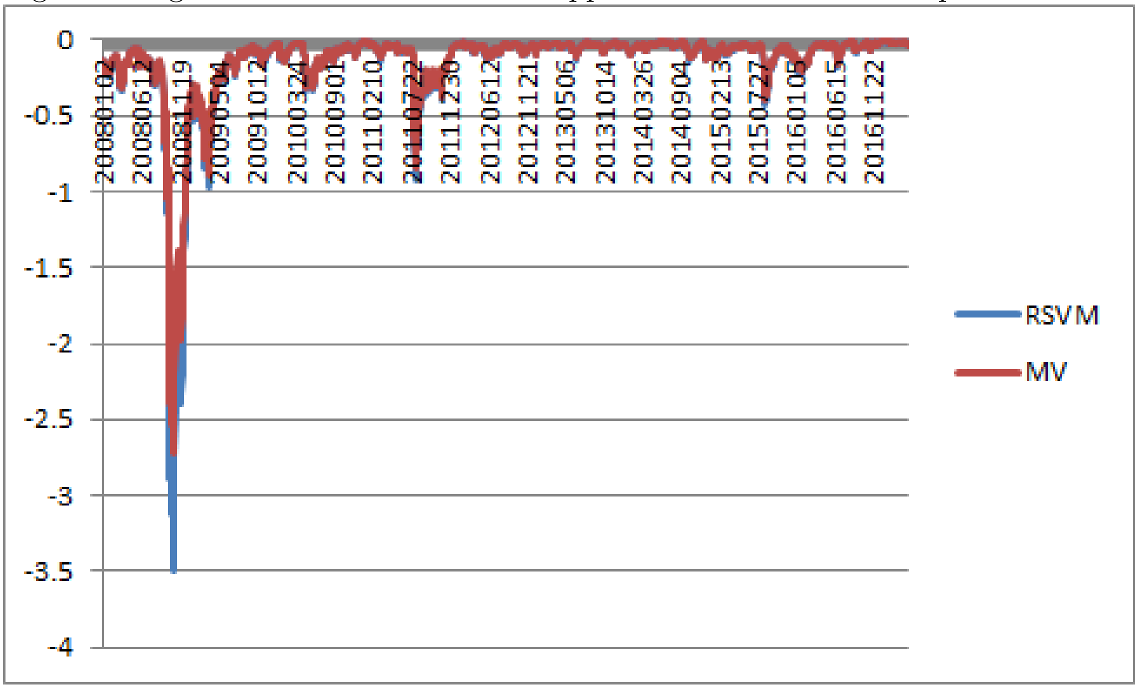


Figure 6: Figure of the RSVM and MV approach over the enire sample for $\alpha = 0.3$

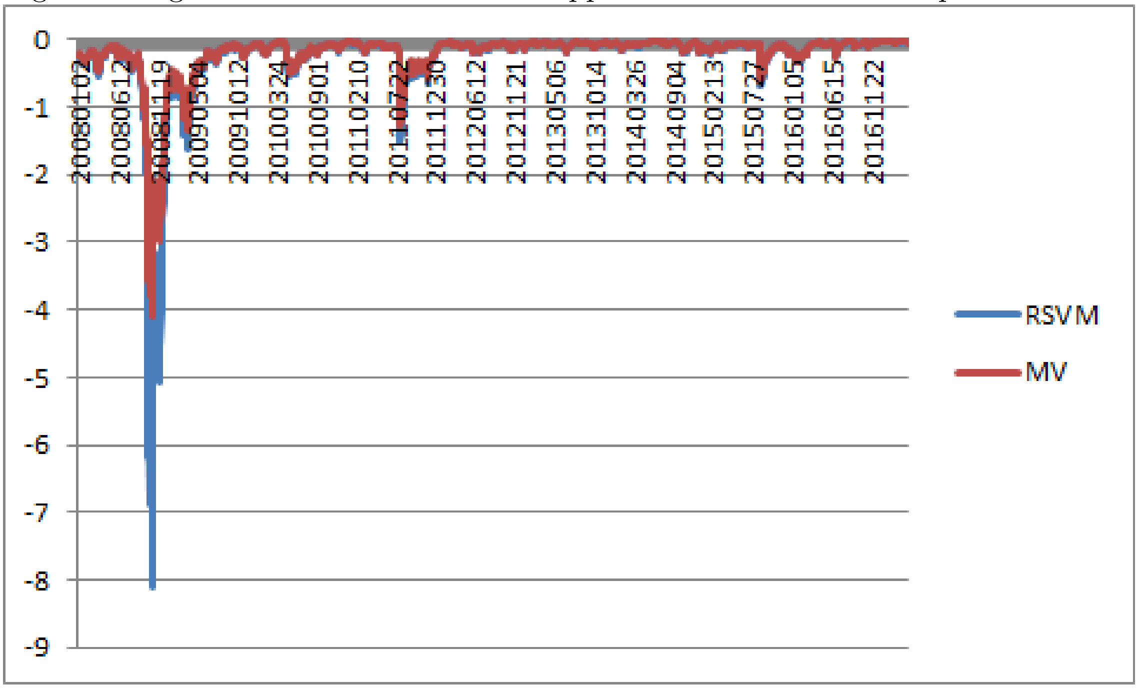


Figure 7: Figure of the RSVM and MV approach over the enire sample for $\alpha = 0.4$

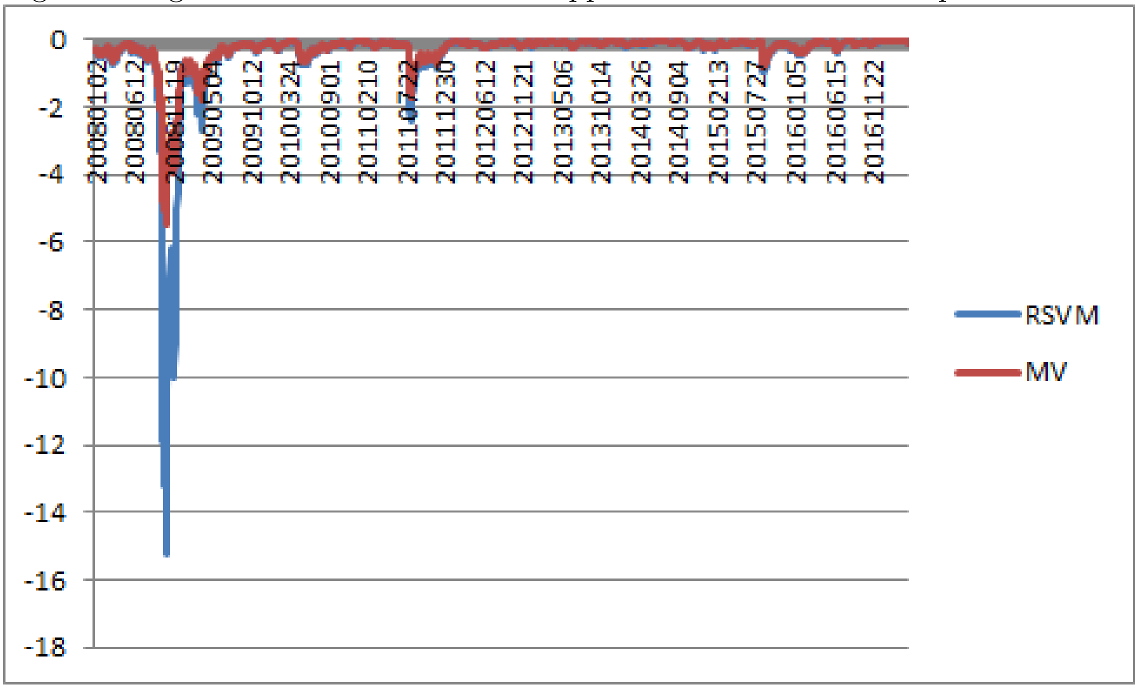


Figure 8: Figure of the RSVM and MV approach over the enire sample for $\alpha = 0.5$

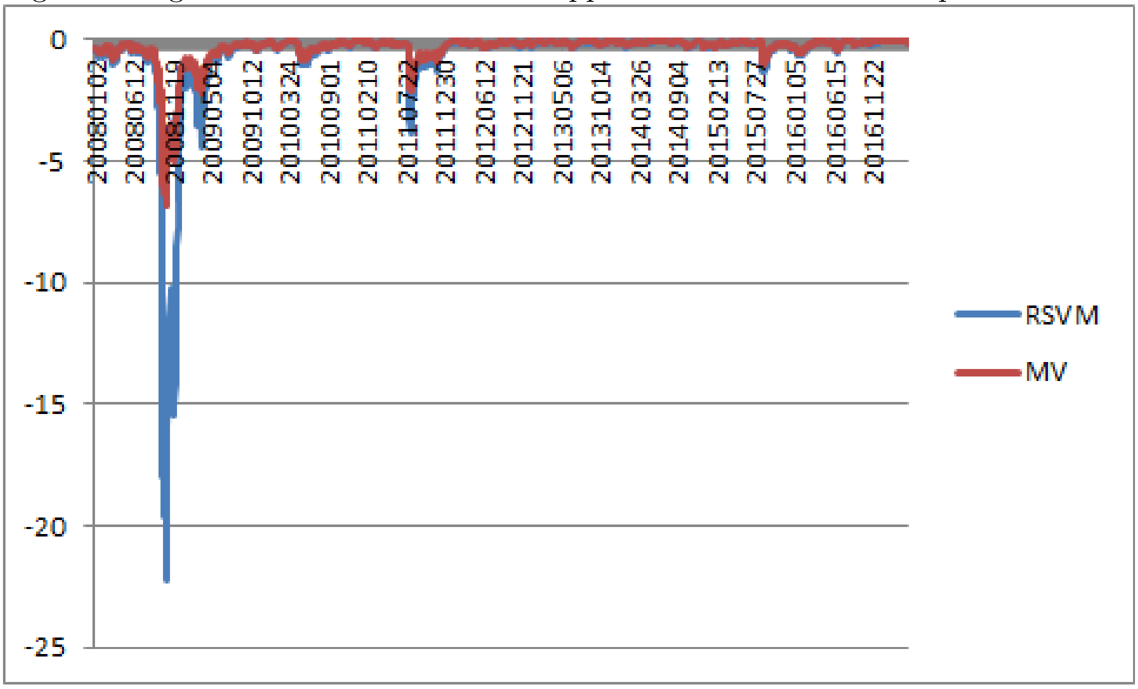


Figure 9: Figure of the RSVM and MV approach in the financila crisis for $\alpha = 0.01$

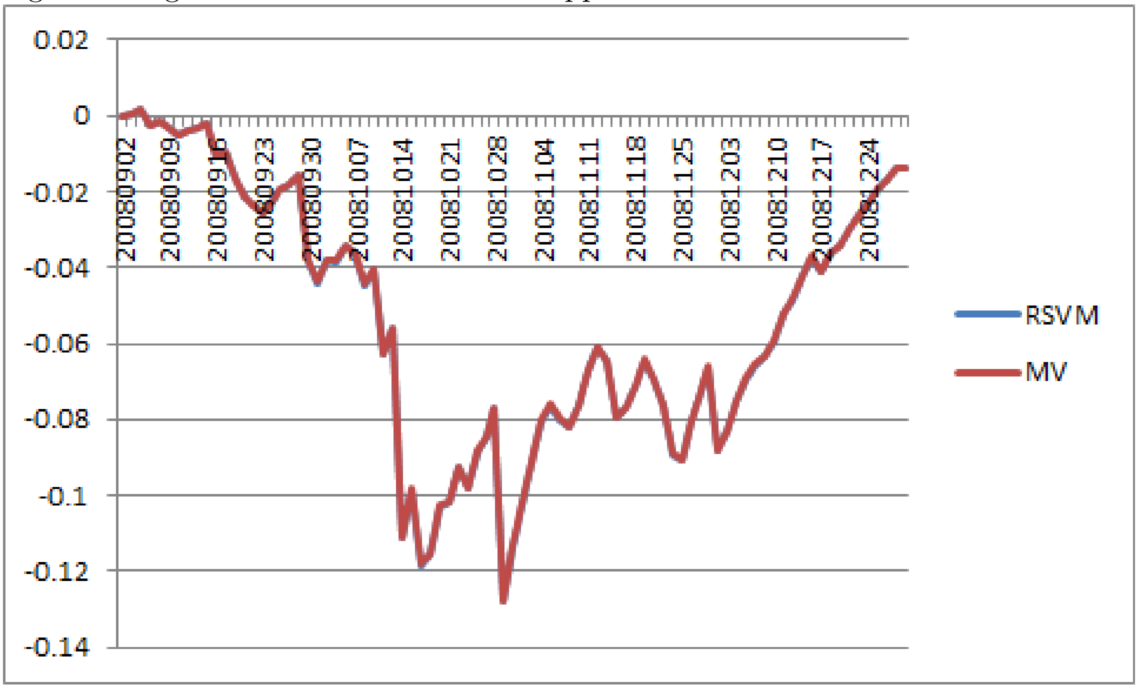


Figure 10: Figure of the RSVM and MV approach in the financila crisis for $\alpha = 0.1$

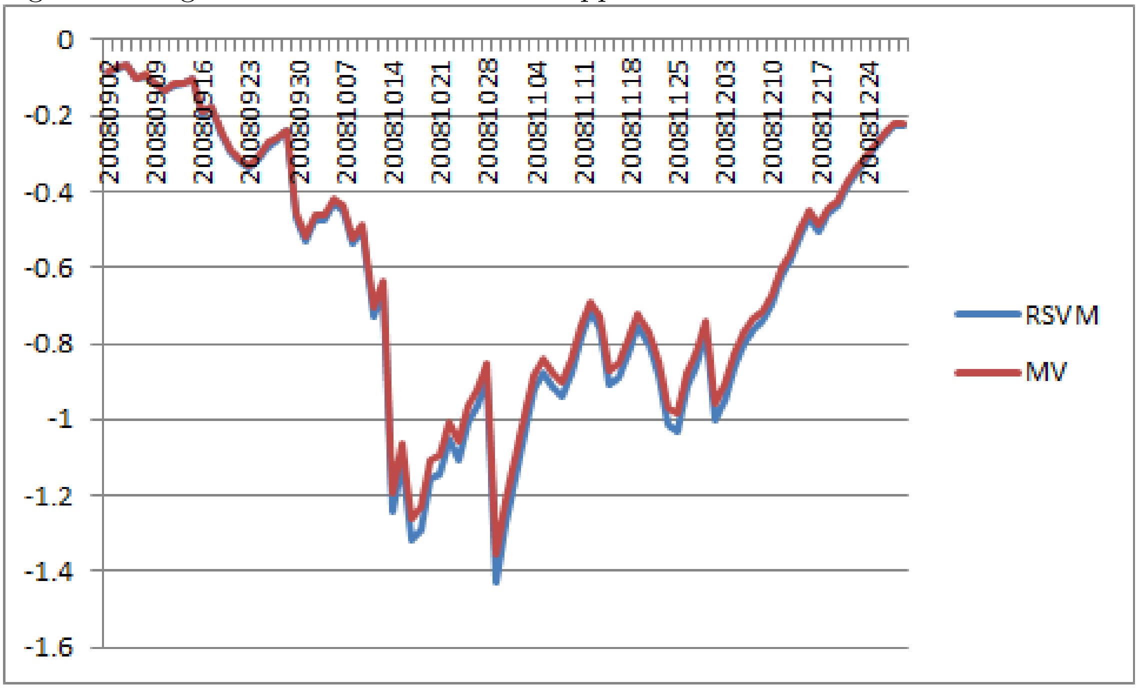


Figure 11: Figure of the RSVM and MV approach in the financila crisis for $\alpha = 0.2$

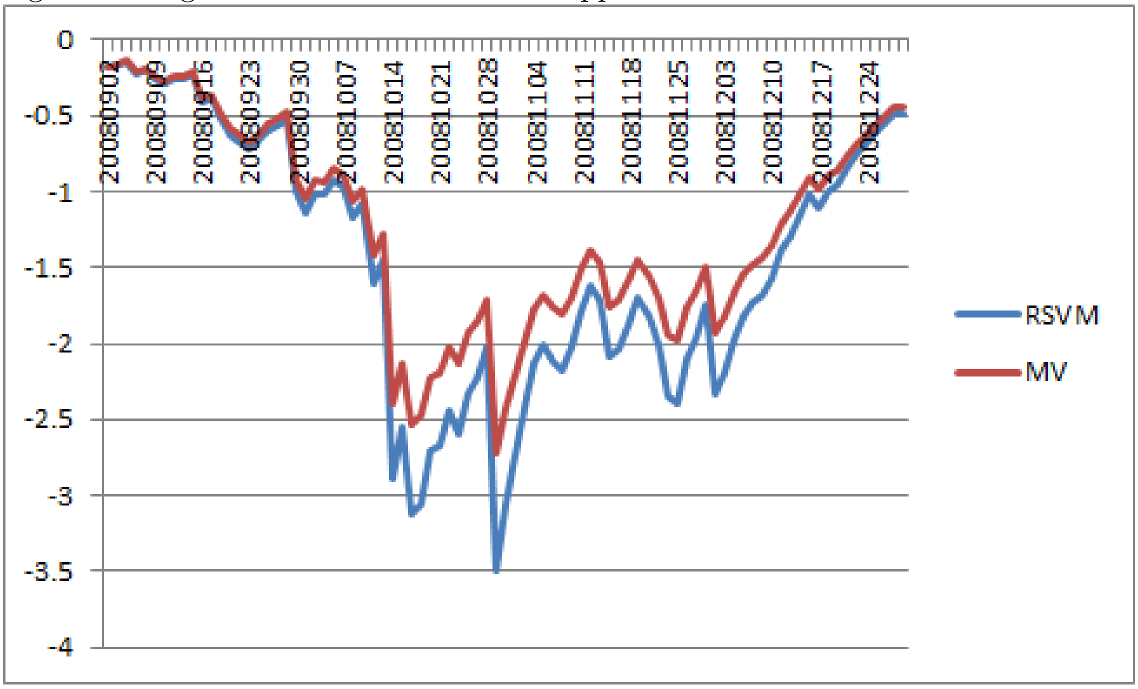


Figure 12: Figure of the RSVM and MV approach in the financila crisis for $\alpha = 0.3$

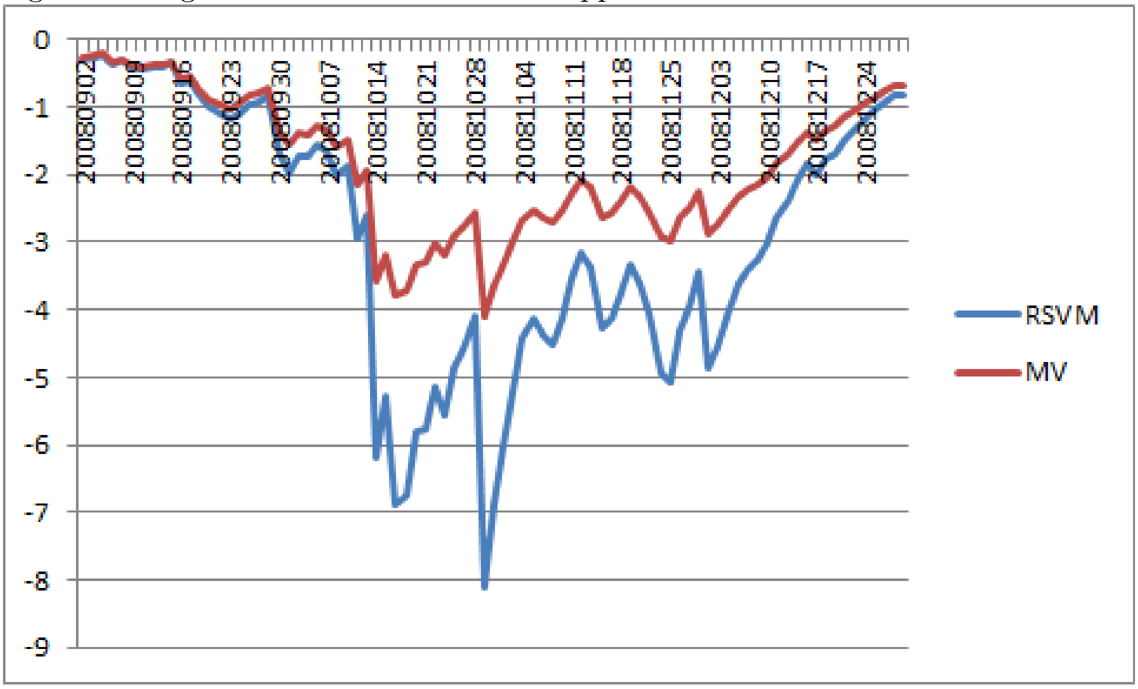


Figure 13: Figure of the RSVM and MV approach in the financila crisis for $\alpha = 0.4$

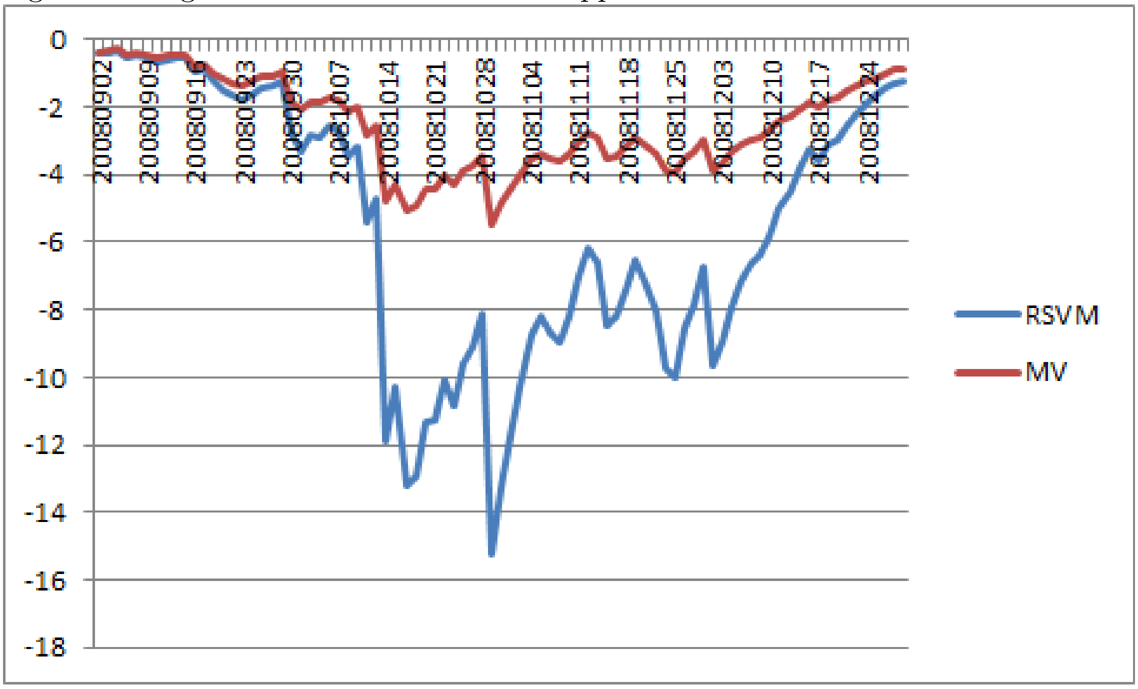
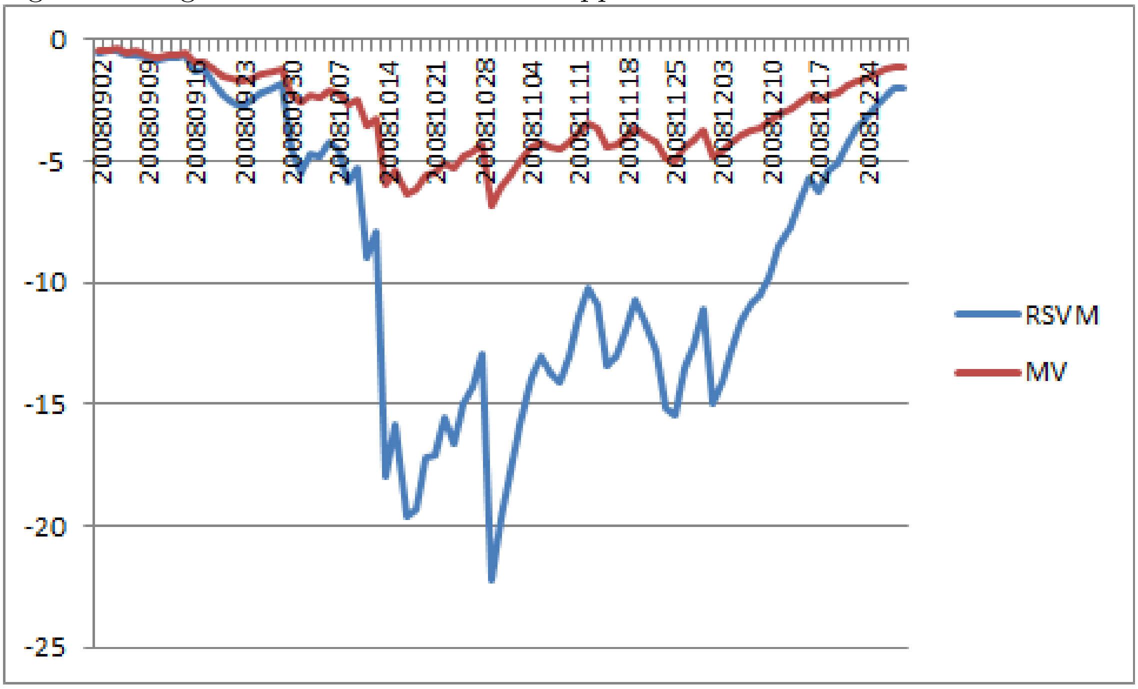


Figure 14: Figure of the RSVM and MV approach in the financila crisis for $\alpha = 0.5$



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