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A Numerical Example of Derivatives Price with Non-cash Collateralization

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Abstract

In this study, we propose a derivatives pricing model where both cash and a non-cash asset are posted as collateral for a derivatives contract. We assume that the participant exchanges the posted non-cash collateral for money through the repo market. Our pricing formula for the collateralized claim is derived taking into account the investment of the received risk-free collateral asset. The resulting pricing formula describes a multi-curve framework and depends on the proportion of cash or non-cash collateral and the repo market haircut. We assume that these parameters are arbitrarily determined by the market. We then carry out a sensitivity analysis and describe how the proportion of the cash collateral and the haircut affect the derivatives price in a constant as well as stochastic interest rate environment. We finally show that the sensitivity analyses results depend on the funding cost.

JEL Classification: G10, G12, G13
Keywords: derivatives pricing, non-cash collateralization, funding costs

1 Introduction

In this study, we propose a derivatives pricing model where both cash and a non-cash asset are posted as collateral for a derivatives contract. Derivatives contracts have applied collateralization to mitigate counterparty risk. That is, if a participant faces a negative marked-to-market (MtM) exposure of a d1erivatives contract, then she/he posts the collateral to her/his counterparty. This collateralization is referred to as variation margin. In recent, it has been suggested that an initial margin is applied so that the collateral collector can hedge the liquidity risk of the posted collaterals if the participant defaults (BCBS 2015). We consider only the variation margin for convenience.

If one considers pricing the collateralized derivatives products, the pricing rule will be different from the one using the risk-free rate as the discount rate. In fact, Xiao (2017) empirically demonstrated that the swap rate quoted in the market reflects the counterparty risk as well as (cash) collateralization. As regards the pricing formula, Johannes and Sundaresan (2007) show that the

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discount rate is the difference between the risk-free rate and the net benefit of receiving cash collateral. When the net benefit is the difference between the risk-free rate and the collateral rate, the claim payoff is discounted using the collateral rate. That is, a discount factor is assigned to the collateral rate to remain in the single-curve setting. This perspective has been supported by Fujii et al. (2010), Piterbarg (2010), and Kan and Pedersen (2012), because the collateral rate is defined as a return to the posted cash collateral especially under a credit support annex (CSA) agreement (see also Fujii and Takahashi 2013, and Gregory 2015). From Fujii et al. (2011), the overnight (ON) rate is used as a collateral rate and its term structure is estimated using Overnight Index Swap (OIS) rate, that is, OIS discounting (Smith 2013).

The collateral rate is used for discounting because cash collateralization uses a single currency. However, most of the clearing houses or security exchanges provide market participants an opportunity to use both cash and non-cash assets as collateral, allowing denomination in multiple currencies. For example, ICE Clear Europe accepts government bonds of various countries (e.g., EU, Japan, the US) issued in different currencies as collateral[1].

1.1 Literature Review

The derivative pricing formula includes multiple interest rates (referred to as multi-curve) when one considers the denominations of the different cash or non-cash collateralization in the derivatives pricing model. Fujii et al. (2010, 2011) and Fujii and Takahashi (2013, 2016) proposed derivatives pricing models with cash collaterals denominated in different currencies, and showed that the claim price is discounted with the collateral rate of not only the local currency, but also the different other currencies. Note that the pricing formula of a claim is derived considering the investment of the posted collateral (i.e., the pricing model by the asset side) in Fujii et al. (2010) and Fujii and Takahashi (2016).

More generally, the derivative pricing model leads to a multi-curve setting by considering funding even without a CSA agreement between the parties. Lou (2015a, b) considered a financial market model without the CSA agreement where a participant with negative exposure posts a cash collateral and earns her/his debt interest rate, to show that the fair value of a swap contract is discounted with the liability-side funding rate. Studying more general situations, Crépey (2015a, b) considered a model where the participants in a derivatives contract source money at the funding costs for hedging or collateralization, to provide a multi-curve framework. Crépey's models can accommodate the borrowing rate, repo rate, and funding rates in different currencies.

The discount factor also depends on some parameters besides multiple interest rates, given that one considers the non-cash collateralization. Constructing a replication strategy for collateralized derivatives, Lou (2017) suggested a pricing method when cash and non-cash assets are posted as collateral, and showed that the discount factor includes the funding rates of participants, a repo rate, and the levels of haircuts besides the risk-free rate. Lou (2017) also carried out sensitivity analyses, and described the effects of collateralization on the valuation adjustments (xVAs). Lou (2017) has two types of haircut, one applied when the participant receives non-cash collateral, the other applied in the repo market. Takino (2018) proposed an equilibrium pricing model for an OTC option when a non-cash asset is posted as collateral, and modeled collateralization from the perspective of the liability side, following Lou (2015a, b). Brigo et al. (2017) proposed an option pricing model with counterparty risk (i.e., the so-called vulnerable option without collateralization) when the participant sources assets for derivatives hedging from the repo market. The pricing formula given by Brigo et al. (2017) also corresponds to the multi-curve framework; that is, the

[1] https://www.theice.com/clear-europe/treasury-and-banking

discount factor includes the funding rate and repo rate. Note that Brigo et al. (2017) and Lou (2017) assumed that all interest rates are constant. Our model is closely related to the Lou's (2017) model. We consider a model with a non-cash asset as collateral and derive a claim pricing formula based on the asset side; we then extend it to a stochastic interest rate environment. We also assume that the collateral payer arbitrarily chooses the proportion of the cash (or non-cash) collateral amount for posting as collateral. In this sense, our model is closely related to Brigo et al.'s (2017) model while the proportion of the cash collateral in Lou (2017) is determined by the haircut for non-cash collateralization. In fact, we carry out a sensitivity analysis of the derivatives prices related to the parameters determined by the participant arbitrarily following Brigo et al. (2017).

1.2 This Study

In this study, we assume that both cash and non-cash assets are used as collateral, and that the collaterals are continuously and perfectly posted in a single currency. We also assume that the collateral receiver sources money by posting the received non-cash collateral asset to the repo market and invests the funded money in a risk-free asset. That is, we consider an asset side pricing, as demonstrated in previous studies (e.g., Fujii and Takahashi 2016), and derive a pricing formula for the collateralized claim from the time-varying invested collateral value. The corresponding products are the collateralized interest rate swap and collateralized option contracts. As regards the swap contract, we derive the swap rate pricing formula after providing the price of the collateralized bond. The discount factor in our formula includes of the difference between the risk-free rate and the net return from investment of the collateral as described above. Since the non-cash asset posted as part of the collateral is exchanged for cash through the repo market, our discount factor includes the collateral rate and repo rate besides the risk-free rate and repo market haircut. It also depends on the proportion of the cash collateral to the total collateral (we call this "cash collateral ratio"). When the collateral is fully posted in cash, the discount factor of our formula coincides with the collateral rate, as shown in previous studies.

We carry out sensitivity analyses, to find the effects of non-cash collateralization on the derivatives pricing based on a pricing formula. Recall that our pricing formula includes the cash collateral ratio and haircut as parameters. These parameters are assumed to be arbitrarily determined by the market participants or clearing houses. Brigo et al. (2017) carried out a sensitivity analysis of the option price with respect to the parameters that the market participant can determine arbitrarily. Hence, following Brigo et al. (2017), we demonstrate the effects of the cash collateral ratio and haircut on derivatives pricing. We consider cases where all the interest rates are constant and stochastically driven. We apply the Vasicek and CIR models to describe stochastically the drives in interest rates. The Vasicek model is an example of negative interest rate models, while the CIR model is an example of non-negative interest rate models.

Our findings are as follows. First, when all interest rates are constant, the cash collateral ratio increases the claim price. The effect of a haircut in the repo market on the claim price differs by the risk-free rate and repo rate. When the repo rate is larger (lower) than the risk-free rate, the haircut increases (decreases) the claim price. These results are interpreted from the perspective of funding cost. In this study, we assume that the collateral collector invests the posted collateral by converting it to money. Thus, the collateral receiver incurs funding costs (i.e., at the collateral and repo rates) and obtains returns from investment. Note that the collateral collector also pays the claim fee (e.g., option price) at the date of contract in our model. Thus, the collateral receiver accepts the higher claim price if she/he can relatively reduce the funding cost, and vise versa.

Second, for the case of stochastically varying interest rates and option contract, we obtain the same results as for the constant rate case. Also, we find no difference in results under the Vasicek and CIR models in the stochastic interest rate environment. These results show that our pricing model is robust. Finally, the sensitivity analysis results for the swap rate are quite the opposite to those obtained for the option case. We can deduce this easily from the relation between the returns and asset price as well as from the derivatives product characteristic.

The remainder of the study is organized as follows. In the next section, we set a financial market model and derive a claim pricing formula. We also conduct a sensitivity analysis of the claim price when all interest rates are constant. In Section 3, we carry out sensitivity analyses of the swap rate and option price under stochastic interest rate environments. Section 4 concludes the study.

2 Model and Pricing

2.1 Collateral Agreement and Payoff with Collateralization

We first set a collateral agreement to derive the pricing formula of derivatives with collateralization. The collateral is continuously and perfectly posted. "Perfectly" posted collaterals means that the collateral amount is equal to the derivatives' MtM value. Furthermore, continuously posting collaterals eliminates the counterparty risk. The collateral assets can include non-cash assets as well as cash. For cash collateral, the collateral receiver pays the collateral rate r^c. If the cash proportion of the total collateral amount (we call this the cash collateral ratio) is η $(0 \leq \eta \leq 1)$ and the MtM value is V_t at time t, then the time-varying payout to the collateral payer is

$$r_t^c \eta V_t dt.$$

The time t value of the posted non-cash collateral automatically becomes

$$(1-\eta)V_t.$$

However, as regards the non-cash collateral, the collateral receiver should pay the collateral payer the returns gained from the non-cash asset posted as collateral. For example, if Dealer A receives a coupon-bearing bond from Bank B, A should pay B the coupon obtained from holding the bond. We, however, assume that a zero-coupon bond is a non-cash collateral. Thus, we effectively assume that the non-cash collateral earns nothing at all. Instead of the posted non-cash collateral having no payment, non-cash collateralization should have no haircut.

Next, we introduce the behavior of the participant who receives the collateral assets. The participant invests the cash collateral in a risk-free asset at the risk-free rate r. A posted non-cash collateral is used to source money from the repo market, and this funded money also is invested in the risk-free asset. We assume that the repo market offers haircut h $(0 \leq h \leq 1)$ to the investors; that is, the participant can obtain money worth only $(1-h) \times 100\%$ of the posted asset. Moreover, the investor funding money through the repo pays an interest, that is, the repo rate r^p. Summarizing the behavior of the collateral collector, we represent the instantaneous change in the received collateral value as

$$y_t V_t dt := \{r_t(\eta + (1-\eta)(1-h)) - (r_t^c \eta + r_t^p (1-\eta)(1-h))\} V_t dt.$$

Following the approach of Fujii and Takahashi (2016), we derive a collateralized claim pricing rule. Under the risk-neutral measure Q, the collateralized claim price $q(t)$ at time t is given by

$$
\begin{aligned}
q(t) =& E_t^Q\left[e^{-\int_t^T r_s ds} q(T) + \int_t^T e^{-\int_t^s r_u du} y_s V_s ds\right] \\
=& E_t^Q\left[e^{-\int_t^T r_s ds} q(T) + \int_t^T e^{-\int_t^s r_u du} y_s q(s) ds\right],
\end{aligned}
$$

assuming that $V_s = q(s)$ for $t \le s \le T$ and full collateralization, where $q(T)$ is the claim payoff at maturity T. If we set

$$X_t = e^{-\int_0^t r_s ds} q(t) + \int_0^t e^{-\int_0^s r_u du} y_s q(s) ds, \tag{2.1}$$

then

$$
\begin{aligned}
E_t^Q[X_T] =& E_t^Q\left[e^{-\int_0^T r_s ds} q(T) + \int_0^T e^{-\int_0^s r_u du} y_s q(s) ds\right] \\
=& e^{-\int_0^t r_s ds} E_t^Q\left[e^{-\int_t^T r_s ds} q(T) + \int_t^T e^{-\int_t^s r_u du} y_s q(s) ds\right] + \int_0^t e^{-\int_0^s r_u du} y_s q(s) ds \\
=& e^{-\int_0^t r_s ds} q(t) + \int_0^t e^{-\int_0^s r_u du} y_s q(s) ds \\
=& X_t.
\end{aligned}
$$

Thus, X_t is a Q-martingale. From (2.1), we have

$$dX_t = -r_t e^{-\int_0^t r_s ds} q(t) dt + e^{-\int_0^t r_s ds} dq(t) + e^{-\int_0^t r_u du} y_t q(t) dt,$$

which yields

$$dq(t) = (r_t - y_t) q(t) dt + e^{\int_0^t r_s ds} dX_t. \tag{2.2}$$

We rewrite (2.2) as

$$-dq(t) = -(r_t - y_t) q(t) dt - e^{\int_0^t r_s ds} dX_t. \tag{2.3}$$

From Proposition 6.2.1 in Pham (2009), the solution of the backward stochastic differential equation (2.3) is given by

$$\beta_t q(t) = E_t^Q[\beta_T q(T)], \tag{2.4}$$

where $\beta_t = \exp\left(-\int_0^t (r_s - y_s) ds\right)$. Equation (2.4) leads to the pricing formula of the collateralized claim $q(T)$ as follows:

Proposition 2.1. *If r is a risk-free rate, r^c is the collateral rate, r^p is the repo rate, and h is the haircut of the repo, then the price of a (continuously and perfectly) collateralized contingent claim under the risk-neutral is given by*

$$q(t) = E_t^Q\left[e^{-\int_t^T (r_s - y_s) ds} q(T)\right], \tag{2.5}$$

where

$$y_s = r_s(\eta + (1-\eta)(1-h)) - (r_s^c \eta + r_s^p (1-\eta)(1-h)). \tag{2.6}$$

Johannes and Sundaresan (2007) called y in (2.6) as the "net benefit", and derived the formula (2.5). We differ with Johannes and Sundaresan (2007) in that our net benefit consists of the repo rate r^p, cash collateral ratio η, and haircut h.

Remark 2.1. *If the collateral is fully posted by cash, that is, $h = 1$, then (2.6) is*

$$y_s = r_s - r_s^c.$$

Thus, the pricing formula (2.5) is given by

$$q(t) = E_t^Q \left[e^{-\int_t^T r_s^c ds} q(T) \right].$$

This agrees with the result in Fujii et al. (2010), for instance.

2.2 Constant Rates Case and Sensitivity Analysis

We consider the case where all the rates are positive and constant during the life of the derivatives contracts; that is, $r_t \equiv r$ (> 0), $r_t^c \equiv r^c$ (> 0), and $r_t^p \equiv r^p$ (> 0) for $0 \leq t \leq T$. This analysis provides an intuition on how the collateral agreement affects the claim price. The pricing formula (2.5) of the claim with maturity T under the constant interest rates is

$$q(t) = e^{-(r-y)(T-t)} E_t^Q \left[q(T) \right], \tag{2.7}$$

where $y = r(\eta + (1-\eta)(1-h)) - (r^c \eta + r^p (1-\eta)(1-h))$.

Assuming that all the interest rates are constant, we carry out the sensitivity analysis of the claim price pertaining to the cash collateral ratio η and haircut h. We assume that $q(t) > 0$ for $0 \leq t \leq T$, which is valid for at least the option cases. We further assume that the payoff function $q(T)$ does not depend on η and h at all. We first observe the effect of cash collateral ratio η on the derivatives price. A partial derivative with respect to η is

$$\frac{\partial q(t)}{\partial \eta} = \{ r^p - (r^p - r)h - r^c \} (T-t) q(t).$$

The sign of the partial derivative depends on several parameters. Because $h < 1$, we approximately set $(r^p - r)h = 0$. Then, we have

$$\frac{\partial q(t)}{\partial \eta} \approx (r^p - r^c)(T-t) q(t).$$

Therefore, an increase in the cash collateral ratio η increases the derivatives price when $r^p > r^c$ and decreases it when $r^p < r^c$. Here, since the ON rate is applied as the collateral rate in practice (Fujii and Takahashi 2016), we expect $r^p > r^c$. Thus, the cash collateral ratio increases the derivatives price.

This result is interpreted as follows: First, the participant evaluating the derivatives pays the derivatives fee to the counterparty and receives collateral from her/him. In this study, the collateral receiver should obtain funds through the repo market at the repo rate for the received non-cash collateral. This means that a derivatives contract with collateralization is costly for the collateral receiver or participant with a positive exposure at the MtM date. Thus, the participant with a positive exposure might be willing to pay less derivatives fee if the cost of funding the received collateral is high. From this perspective, we can interpret the effects of the cash collateral ratio η

on the derivatives price. An increase in cash collateral ratio reduces the amount funded through the repo market. Now, since the relation of the funding costs is given by

$$\text{Repo rate } r^p > \text{Collateral rate } r^c,$$

the participant can obtain a cash collateral with low funding cost. In other words, by spending the money funded from the posted collateral on the derivatives contract, the participant can enter the derivatives contract with a low funding cost. Hence, the participant with a positive exposure at the time of contract accepts a higher price.

Next, we consider the effect of a haircut in the repo market on the derivatives price. A partial derivative with respect to h is

$$\frac{\partial q(t)}{\partial h} = (r^p - r)(1-\eta)(T-t)q(t).$$

Thus, for a given η $(0 \leq \eta \leq 1)$, an increase in haircut h increases the derivatives price if $r^p > r$ and decreases it if $r^p < r$. The result is interpreted as follows: The increase in haircut reduces the amount funded through repo. This effectively reduces the cost of funding (i.e., the repo rate $\times$ the borrowed money amount via repo), and the collateral receiver (i.e., the participant with positive exposure) might accept the higher claim price. However, when $r > r^p$, the participant obtains a higher return from investing the money funded through collateralization. This is thus supposed to decrease the claim price from the relation between the return from and price of the assets. However, when $r < r^p$, that is, when the collateral receiver cannot earn relatively more money since the repo cost is larger than the return, the effective reduction in funding cost from the increase in haircut increases the derivatives price.

Note that these sensitivity analyses consider the derivatives products where prices are presented in (2.5) like an option. We use numerical simulation for the sensitivity analysis of a swap rate because the swap rate does not (2.5).

3 Numerical Results

We consider the risk-neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, Q)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration generated by a four-dimensional standard Brownian motion, $W = (W_1, \ldots, W_4)$. We also consider a collateralized interest rate swap for forward LIBOR and a collateralized option written on general asset S as examples of derivatives.

3.1 Collateralized Zero-coupon Bond

We first consider the price of a collateralized zero-coupon bond to derive the swap rate. We denote the price of the collateralized zero-coupon bond at time t as $P^c(t,T)$; that is,

$$P^c(t,T) = E_t^Q\left[e^{-\int_t^T (r_s - y_s)ds}\right].$$

3.2 Collateralized Interest Rate Swap

We consider a collateralized swap contract for a three-month LIBOR, where the terminal is one year, for example. The term “collateralized” in the swap contract implies that the future swap values are discounted with the difference between the risk-free rate and net benefit. Technically, the LIBOR rate is solved with the collateralized zero-coupon bond. The settlement of the swap contract is executed every three months, with the contract expiring at one year later. We denote the three-month LIBOR rate for $[T_{j-1}, T_j]$ at time $t < T_{j-1}$ as $L(t; T_{j-1}, T_j)$, and the pay/receive date as $\{T_1, \dots, T_4\}$, where T_4 is the expire date. We also denote the collateralized swap rate by R^c; now, from (2.5), the the present value V_0 of the swap contract is

$$V_0 = \sum_{j=1}^{4} E^Q \left[e^{-\int_0^{T_j} (r_s - y_s) ds} \delta_j (R^c - L(T_{j-1}; T_{j-1}, T_j)) \right], \tag{3.1}$$

where $\delta_j = T_j - T_{j-1}$ and $\delta_j = 0.25$ for all j. From (3.1), we have

$$\begin{aligned}
V_0 &= \sum_{j=1}^{4} E^Q \left[e^{-\int_0^{T_j} (r_s - y_s) ds} \delta_j (R^c - L(T_{j-1}; T_{j-1}, T_j)) \right] \\
&= \sum_{j=1}^{4} \left\{ E^Q \left[e^{-\int_0^{T_j} (r_s - y_s) ds} \right] \delta_j R^c - E^Q \left[e^{-\int_0^{T_j} (r_s - y_s) ds} \delta_j L(T_{j-1}; T_{j-1}, T_j) \right] \right\} \\
&= \sum_{j=1}^{4} \left\{ P^c(0, T_j) \delta_j R^c - P^c(0, T_j) \delta_j E^{T_j} \left[L(T_{j-1}; T_{j-1}, T_j) \right] \right\} \\
&= \sum_{j=1}^{4} \left\{ P^c(0, T_j) \delta_j R^c - P^c(0, T_j) \delta_j L(0; T_{j-1}, T_j) \right\} \\
&= \sum_{j=1}^{4} \left\{ P^c(0, T_j) \delta_j R^c - P^c(0, T_j) \left(\frac{P^c(0, T_{j-1})}{P^c(0, T_j)} - 1 \right) \right\} \\
&= \sum_{j=1}^{4} \left\{ P^c(0, T_j) \delta_j R^c - (P^c(0, T_{j-1}) - P^c(0, T_j)) \right\},
\end{aligned} \tag{3.2}$$

where E^{T_j} is the expectation under the T_j-forward measure. The collateralized LIBOR rate is given by $L(t; T_{j-1}, T_j) = \frac{1}{\delta_j} \left(\frac{P^c(t, T_{j-1})}{P^c(t, T_j)} - 1 \right)$ for $t < T_{j-1}$; we apply this from the forth line to the fifth line in (3.2).

Since the swap rate equals the swap value zero in (3.2), that is, $V_0 = 0$, the swap rate is given by

$$R^c = \frac{P^c(0, T_0) - P^c(0, T_4)}{\sum_{j=1}^{4} \delta_j P^c(0, T_j)}. \tag{3.3}$$

We set $T_0 = 0.25$, $T_1 = 0.50$, $T_2 = 0.75$, $T_3 = 1.00$, and $T_4 = 1.25$ in the following numerical implementations.

3.3 Collateralized Option

We also consider an option contract written on the general asset price S in (3.5). For example, we consider a plain vanilla call with maturity T; that is, the payoff function is $q(T) = \max(S_T - K, 0)$. From (2.5), the collateralized option price $q(0)$ is given by

$$q(0) = E^Q \left[e^{-\int_0^{T} (r_s - y_s) ds} q(T) \right]. \tag{3.4}$$

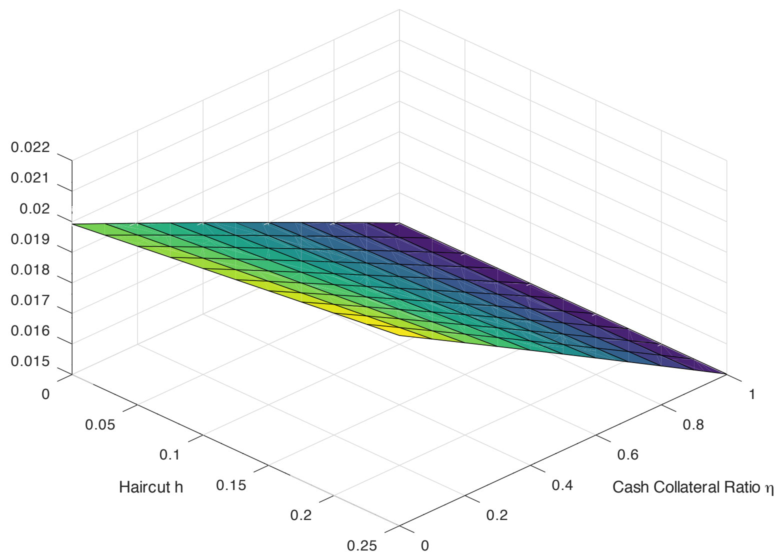

Figure 1: Collateralized swap rate under the Vasicek model (3.5) when $r_0 > r_0^p$.

We assume that the dynamics of S is driven by

$$dS_t = S_t(r_t dt + \sigma_S dW_{4t}), \quad S_0 = S.$$

We set $S_0 = 22865$, $\sigma_S = 0.20$, $K = 22750$, and $T = 0.25$ through the paper.

3.4 Vasicek-type Case

Under the Q-measure, we set the risk-free rate, collateral rate, and repo rate as follows:

$$\begin{array}{ll} dr_t = \kappa(\bar{r} - r_t)dt + bdW_{1t}, & r_0 = r, \\ dr_t^c = \kappa_c(\bar{r^c} - r_t^c)dt + b_c dW_{2t}, & r_0^c = r^c, \\ dr_t^p = \kappa_p(\bar{r^p} - r_t^p)dt + b_p dW_{3t}, & r_0^p = r^p, \end{array} \tag{3.5}$$

respectively, where all $\kappa_\cdot$, $\bar{r^\cdot}$, and $b_\cdot$ are constant. Model (3.5) gives the bond price $P^c(t,\cdot)$ (automatically, the swap rate R^c) and the option price $q(0)$ by a closed formula (see Appendix A, B). The fundamental parameters are $\kappa = \kappa_c = \kappa_p = 1.0$, $b = 0.03$, $b_c = 0.01$, $b_p = 0.03$, and $\bar{r^c} = 0.015$.

3.4.1 Collateralized Interest Rate Swap

Figures 1 and 2 plot the results of swap rate R^c in (3.3). These figures correspond to the cases of $r_0 > r_0^p$ ($r_0 = \bar{r} = 0.025$, $r_0^c = \bar{r^c} = 0.015$, $r_0^p = \bar{r^p} = 0.02$) and $r_0^p > r_0$ ($r_0 = \bar{r} = 0.02$, $r_0^c = \bar{r^c} = 0.015$, $r_0^p = \bar{r^p} = 0.03$), respectively. From Figure 1, the haircut increases and the cash collateral ratio decreases the swap rate when $r_0 > r_0^p$. In contrast, Figure 2 demonstrates that the

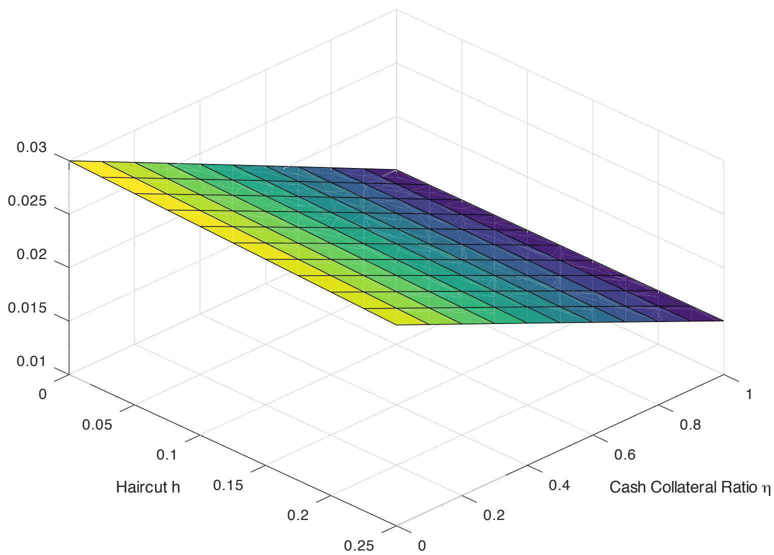

Figure 2: Collateralized swap rate under the Vasicek model (3.5) when $r_0^p > r_0$.

haircut as well as the cash collateral ratio decreases the swap rate when $r_0 < r_0^p$. These results contradict those obtained for the asset price in Section 2.2, but are straightforward from the relation between the return from and price of the asset when one regards the swap rate as a return.

3.4.2 Collateralized Option

Figures 3 and 4 plot the option price $q(0)$ in (3.4). These figures correspond to the cases of $r_0 > r_0^p$ ($r_0 = \bar{r} = 0.025$, $r_0^c = \bar{r^c} = 0.015$, $r_0^p = \bar{r^p} = 0.02$) and $r_0^p > r_0$ ($r_0 = \bar{r} = 0.02$, $r_0^c = \bar{r^c} = 0.015$, $r_0^p = \bar{r^p} = 0.03$), respectively. From Figure 3, the haircut decreases price and the cash collateral ratio increases the option price when $r_0 > r_0^p$. In contrast, Figure 4 shows that the haircut increases the swap rate while the cash collateral ratio increases the option price when $r_0 < r_0^p$. These results agree with those under the constant rates case examined in Section 2.2.

3.5 Cox-Ingersoll-Ross (CIR)-type Case

Under the Q-measure, we set the risk-free rate, collateral rate, and repo rate as

$$\begin{array}{ll} dr_t = \kappa(\bar{r} - r_t)dt + \sigma\sqrt{r_t}dW_{1t}, & r_0 = r, \\ dr_t^c = \kappa_c(\bar{r^c} - r_t^c)dt + \sigma_c\sqrt{r_t^c}dW_{2t}, & r_0^c = r^c, \\ dr_t^p = \kappa_p(\bar{r^p} - r_t^p)dt + \sigma_p\sqrt{r_t^p}dW_{3t}, & r_0^p = r^p, \end{array} \tag{3.6}$$

respectively, where $\kappa_\cdot$, $\bar{r^\cdot}$, and $\sigma_\cdot$ are constant.

In this case, we implement the pricing rules for the swap rate and option using Monte-Carlo simulation. We apply the so-called implicit Eular-Maruyama Scheme in discretization proposed

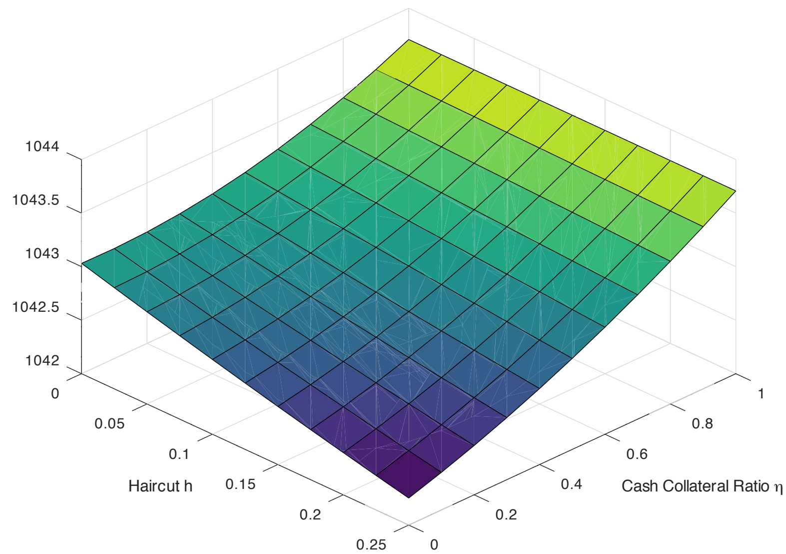

Figure 3: Collateralized option price under the Vasicek model (3.5) when $r_0 > r_p$.

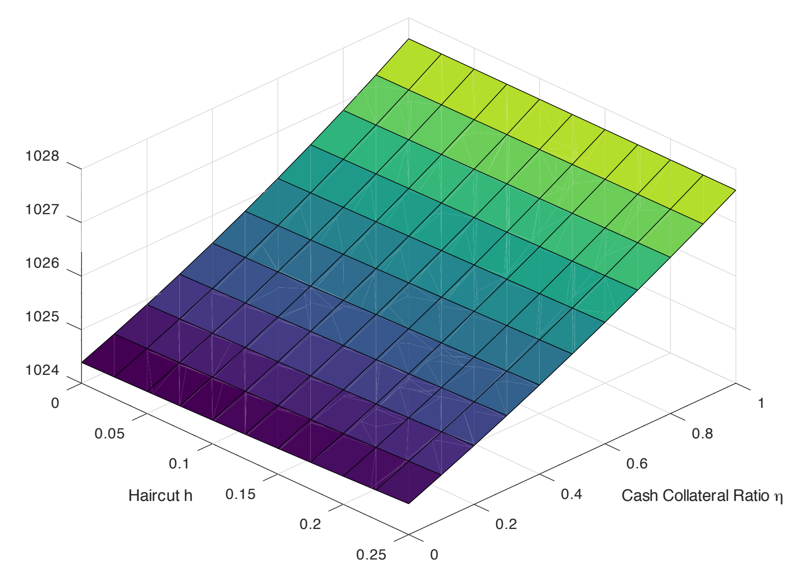

Figure 4: Collateralized option price under the Vasicek model (3.5) when $r_p > r_0$.

Parameter	$r_0 > r_0^p$ case	$r_0 < r_0^p$ case
r_0	0.025	0.025
κ	1.500	1.500
$\bar{r}$	0.025	0.025
σ	0.130	0.125
r_0^c	0.015	0.015
κ_c	1.000	1.000
$\bar{r^c}$	0.015	0.015
σ_c	0.100	0.100
r_0^p	0.020	0.030
κ_p	1.500	1.500
$\bar{r^p}$	0.020	0.030
σ_p	0.125	0.150

Table 1: Parameters for the CIR type case.

by Alfonsi (2013) for processes r, r^c, and r^p (see Appendix C), and use the Milstein scheme (c.f. Gatheral 2006) for process S. We divide one year into 1,000 grids and set the simulation times as 200,000.

In order to ensure a strong convergence of $\hat{Y}$ with order 1 (Alfonsi 2013), we assume that

$$\sigma_{\cdot}^2 < \kappa_{\cdot}\bar{r}_{\cdot}, \quad 1 < \frac{4}{3}\frac{\kappa_{\cdot}\bar{r^{\cdot}}}{\sigma_{\cdot}^2}. \tag{3.7}$$

A parameter set in Table 1 satisfies the convergence condition (3.7).

3.5.1 Collateralized Interest Rate Swap

Figures 5 and 6 plot the results of swap rate R^c in (3.3). These figures correspond to the cases of $r_0 > r_0^p$ ($r_0^p = \bar{r^p} = 0.02$, $\sigma_p = 0.125$) and $r_0^p > r_0$ ($r_0^p = \bar{r^p} = 0.03$, $\sigma_p = 0.15$), respectively. From Figure 5, the haircut increases and the cash collateral ratio decreases the swap rate. In contrast, Figure 6 shows that the haircut as well as the cash collateral ratio decreases the swap rate. These results agree with those obtained for the Vasicek-type case.

3.5.2 Collateralized Option

Figures 7 and 8 plot the results of option price $q(0)$ in (3.4). These figures correspond to the cases of $r_0 > r_0^p$ ($r_0^p = \bar{r^p} = 0.02$, $\sigma_p = 0.125$ in (3.6)) and $r_0^p > r_0$ ($r_0^p = \bar{r^p} = 0.03$, $\sigma_p = 0.15$ in (3.6)), respectively. From Figure 7, the haircut decreases and the cash collateral ratio increases the option price. In contrast, Figure 8 shows that the haircut as well as the cash collateral ratio increases the option price. These results agree with those obtained for the Vasicek-type case (automatically, the constant rates case).

4 Summary

In this study, we proposed a derivatives pricing model with non-cash collateralization. Our model allows for the participant to (re)invest the posted collaterals, and especially, to convert the posted

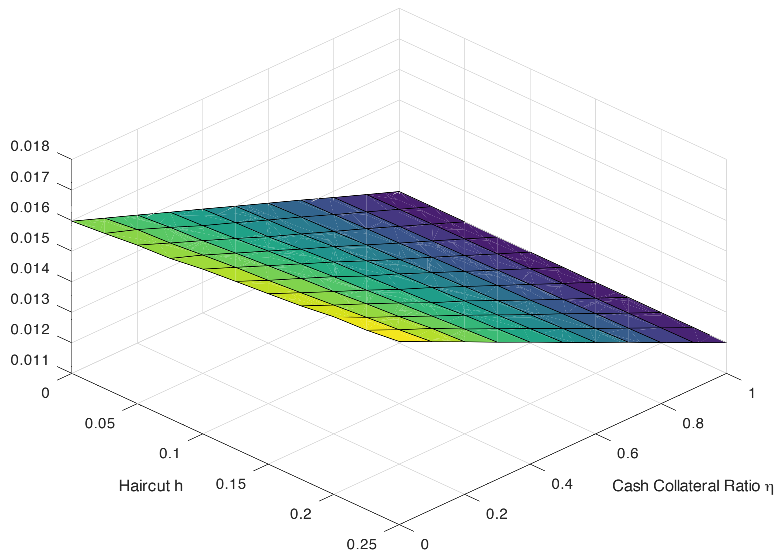

Figure 5: Collateralized swap rate under the CIR model (3.6) when $r_0 > r_0^p$.

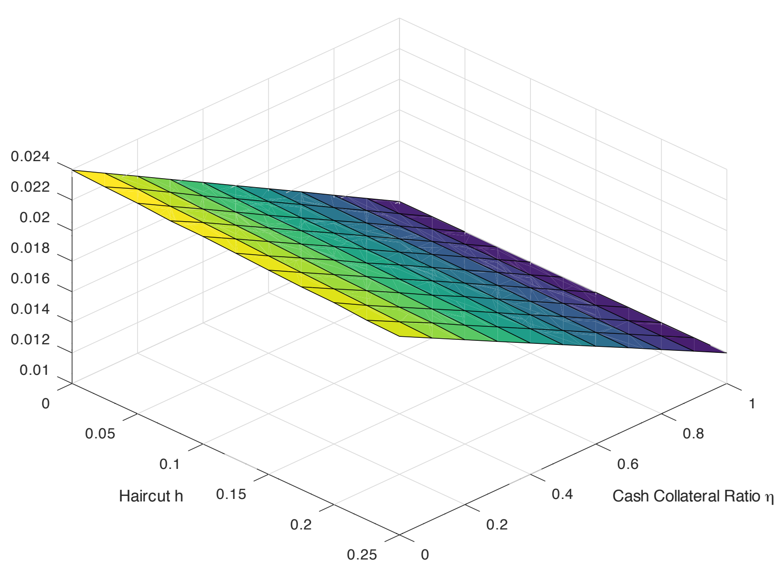

Figure 6: Collateralized swap rate under the CIR model (3.6) when $r_0^p > r_0$.

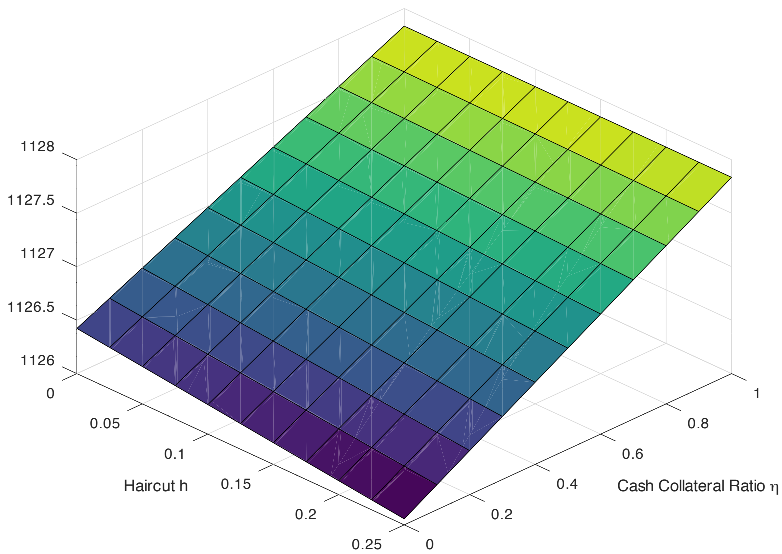

Figure 7: Collateralized option price under the CIR model (3.5) when $r_0 > r_p$.

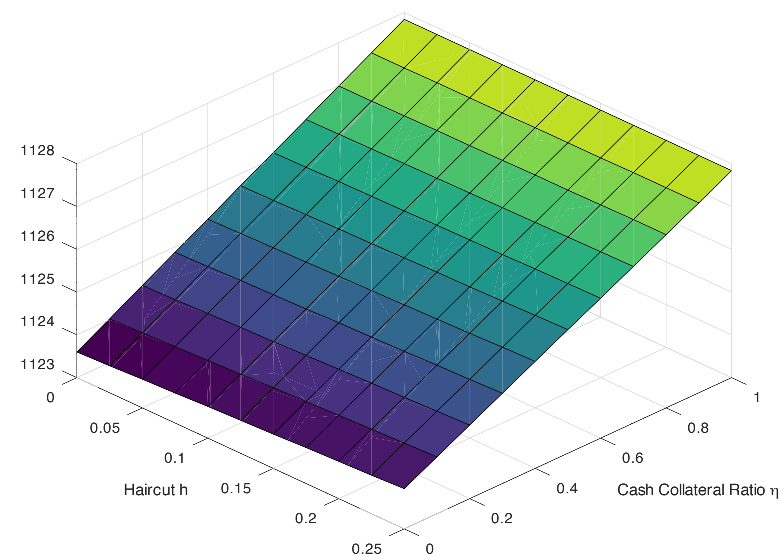

Figure 8: Collateralized option price under the CIR model (3.5) when $r_p > r_0$.

non-cash collateral into money through the repo market. We also assumed that the participant pricing the derivatives products is a collateral receiver or an investor with a positive exposure in derivatives contracts. That is, we proposed an asset-side pricing model and derived a pricing rule for the collateralized claim by considering the investment of the received collateral. The resulting pricing formula describes a multi-curve framework and depends on the combination of a cash and non-cash collateral and the haircut of the repo market. Then, we investigated how the proportion of cash or non-cash collateral to the total posted collateral amount (we call this the cash collateral ratio) and the haircut in the repo market respectively affect the collateralized swap rate and option price. These parameters have different effects on the swap rate and option price. This result is straightforward from the relation between the return and asset price. Changes in the cash collateral ratio and haircut affects the derivatives prices. These results are interpreted from the funding cost or net benefit perspective. In this study, the collateral collector evaluates the derivatives price at the contract date. Hence, when a change in parameter imposes a higher funding cost on the collateral receiver, she/he does not accept the higher derivatives price and vice versa. We also considered the market environments where all the interest rates are constant and vary stochastically. The results of sensitivity analyses are the same for both cases. This shows the robustness of our pricing model.

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A Closed Formula for Collateralized Zero-coupon Bond Price (Section 3.2)

For the interest rate model (3.5), we derive a closed pricing formula for the collateralized zero-coupon bond by following Tabata (2002).

We first set

$$r_t - y_t = F_1 r_t + \eta r_t^c + F_2 r_t^p$$

for $0 \le t \le T$, where

$$\begin{aligned} F_1 =& 1 - (\eta + F_2), \\ F_2 =& (1-\eta)(1-h). \end{aligned}$$

Then, we have

$$\begin{aligned} I(s,t) :=& \int_s^t (r_u - y_u) du \\ =& F_1 \int_s^t r_u du + \eta \int_s^t r_u^c du + F_2 \int_s^t r_u^p du \\ =& F_1 I_r(s,t) + \eta I_c(s,t) + F_2 I_p(s,t) \end{aligned} \tag{A.1}$$

for $s \le t$, where

$$\begin{aligned} I_r(s,t) =& (t-s)\bar{r} + \frac{1}{\kappa}(1 - e^{-\kappa(t-s)})(r_s - \bar{r}) + \frac{b}{\kappa}\int_s^t (1 - e^{-\kappa(t-u)}) dW_{1u}, \\ I_c(s,t) =& (t-s)\bar{r^c} + \frac{1}{\kappa_c}(1 - e^{-\kappa_c(t-s)})(r_s^c - \bar{r^c}) + \frac{b_c}{\kappa_c}\int_s^t (1 - e^{-\kappa_c(t-u)}) dW_{2u}, \\ I_p(s,t) =& (t-s)\bar{r^p} + \frac{1}{\kappa_p}(1 - e^{-\kappa_p(t-s)})(r_s^p - \bar{r^p}) + \frac{b_p}{\kappa_p}\int_s^t (1 - e^{-\kappa_p(t-u)}) dW_{3u}. \end{aligned}$$

The time-t value of the collateralized zero-coupon bond with maturity T is given by

$$\begin{aligned} P^c(t,T) =& E_t^Q \left[e^{-\int_t^T (r_s - y_s) ds} \right] \\ =& E_t^Q \left[e^{-I(t,T)} \right]. \end{aligned} \tag{A.2}$$

Since we use the Vasicek model, we deduce that $I(s,t)$ follows by a normal distribution. Thus, (A.2) is given by

$$P^c(t,T) = \exp\left(-E_t^Q[I(t,T)] + \frac{1}{2} Var_t^Q[I(t,T)] \right).$$

Finally, we provide the conditional expectation and variance of I as follows:

$$E_s^Q[I(s,t)] = F_1 m(r_s, t-s) + \eta m_c(r_s^c, t-s) + F_2 m_p(r_s^p, t-s),$$

where

$$\begin{aligned} m(r_s, \tau) =& \tau\bar{r} + \frac{1}{\kappa}(1 - e^{-\kappa\tau})(r_s - \bar{r}), \\ m_c(r_s^c, \tau) =& \tau\bar{r^c} + \frac{1}{\kappa_c}(1 - e^{-\kappa_c\tau})(r_s^c - \bar{r^c}), \\ m_p(r_s^p, \tau) =& \tau\bar{r^p} + \frac{1}{\kappa_p}(1 - e^{-\kappa_p\tau})(r_s^p - \bar{r^p}); \end{aligned}$$

and

$$Var_s^Q[I(s,t)] = (F_1)^2 v(t-s) + \eta^2 v_c(t-s) + (F_2)^2 v_p(t-s),$$

where

$$\begin{aligned}
v(\tau) =& \frac{b^2}{2\kappa^3}(4e^{-\kappa\tau} - e^{-2\kappa\tau} + 2\kappa\tau - 3), \\
v_c(\tau) =& \frac{b_c^2}{2\kappa_c^3}(4e^{-\kappa_c\tau} - e^{-2\kappa_c\tau} + 2\kappa_c\tau - 3), \\
v_p(\tau) =& \frac{b_p^2}{2\kappa_p^3}(4e^{-\kappa_p\tau} - e^{-2\kappa_p\tau} + 2\kappa_p\tau - 3).
\end{aligned}$$

B Closed Formula for Collateralized Option Price (Section 3.3)

We derive a closed formula for (3.4) by following Kim (2002). We also use the same notations as Kim (2002).

We set

$$Z_T = e^{-\int_0^T (r_t - y_t)dt}(S_T - K).$$

Then, for the dynamics in (3.5), we have

$$\begin{aligned}
Z_T =& e^{-\int_0^T (r_t - y_t)dt}\left(S_0 e^{\int_0^T r_t dt - \frac{1}{2}\sigma_S^2 T + \sigma_S W_{4T}} - K\right) \\
=& S_0 e^{\int_0^T y_t dt - \frac{1}{2}\sigma_S^2 T + \sigma_S W_{4T}} - e^{-I(0,T)}K,
\end{aligned}$$

where $I(\cdot,\cdot)$ is defined in (A.1). By defining

$$\begin{aligned}
X_{1T} =& -(\eta + F_2)\frac{b}{\kappa}\int_0^T (1 - e^{-\kappa(T-u)})dW_{1u} + \eta\frac{b_c}{\kappa_c}\int_0^T (1 - e^{-\kappa_c(T-u)})dW_{2u} \\
& + F_2\frac{b_p}{\kappa_p}\int_0^T (1 - e^{-\kappa_p(T-u)})dW_{3u} + \sigma_S W_{4T},
\end{aligned}$$

$$\begin{aligned}
X_{2T} =& F_1\frac{b}{\kappa}\int_0^T (1 - e^{-\kappa(T-u)})dW_{1u} + \eta\frac{b_c}{\kappa_c}\int_0^T (1 - e^{-\kappa_c(T-u)})dW_{2u} \\
& + F_2\frac{b_p}{\kappa_p}\int_0^T (1 - e^{-\kappa_p(T-u)})dW_{3u}, \\
B_1(T) =& (\eta + F_2)m(r_0, T) - \eta m_c(r_0^c, T) - F_2 m_p(r_0^p, T) \\
B_2(T) =& F_1 m(r_0, T) + \eta m_c(r_0^c, T) + F_2 m_p(r_0^p, T)
\end{aligned}$$

where F_1, F_2, $m(\cdot,\cdot)$, $m_c(\cdot,\cdot)$, and $m_p(\cdot,\cdot)$ are as defined in Appendix A. Now, Z_T is rewritten as

$$Z_T = S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T) + X_{1T}} - K e^{-B_2(T) - X_{2T}}.$$

We further set

$$\Sigma_{11}^T = (\eta + F_2)^2 v(T) + \eta^2 v_c(T) + (F_2)^2 v_p(T) + \sigma_S^2 T,$$

$$\Sigma_{12}^T = \Sigma_{21}^T = -(\eta + F_2)F_1 v(T) + \eta^2 v_c(T) + (F_2)^2 v_p(T),$$

$$\begin{aligned}
\Sigma_{22} =&(F_1)^2 \frac{b^2}{\kappa^2}\int_0^T (1-e^{-\kappa(T-u)})^2 du + \eta^2 \frac{b_c^2}{\kappa_c^2}\int_0^T (1-e^{-\kappa_c(T-u)})^2 du + (F_2)^2 \frac{b_p^2}{\kappa_p^2}\int_0^T (1-e^{-\kappa_p(T-u)})^2 du \\
=&(F_1)^2 v(T) + \eta^2 v_c(T) + (F_2)^2 v_p(T),
\end{aligned}$$

where $v(\cdot)$, $v_c(\cdot)$, and $v_p(\cdot)$ are defined in Appendix A. Then, it holds that

$$Z_T = S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T) + X_{1T}} - K e^{-B_2(T) - X_{2T}},$$

where

$$\begin{pmatrix} X_{1T} \\ X_{2T} \end{pmatrix} \sim \mathcal{N}_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11}^T & \Sigma_{12}^T \\ \Sigma_{21}^T & \Sigma_{22}^T \end{pmatrix} \right].$$

At this point, the inequality of $Z_T \geq 0$ is equivalent to

$$X_{1T} + X_{2T} \geq C(T),$$

where

$$C(T) = -\ln \frac{S_0}{K} - B_1(T) - B_2(T) + \frac{1}{2}\sigma_S^2 T.$$

Therefore, the price of collateralized option G is given by

$$\begin{aligned}
V_0 =&E^Q[\max(Z_T, 0)] \\
=&E^Q\left[\left\{S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T) + X_{1T}} - K e^{-B_2(T) - X_{2T}}\right\} 1_{Z_T \geq 0}\right] \\
=&E^Q\left[S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T) + X_{1T}} 1_{X_{1T} + X_{2T} \geq C(T)}\right] - E^Q\left[K e^{-B_2(T) - X_{2T}} 1_{X_{1T} + X_{2T} \geq C(T)}\right] \\
=:&(\text{I}) - (\text{II}).
\end{aligned} \tag{B.1}$$

We solve terms (I) and (II) in (B.1) using the result in Kunitomo and Takahashi (1992) as applied by Kim (2002).

$$\begin{aligned}
(\text{I}) =&S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T)} E^Q\left[e^{X_{1T}} 1_{X_{1T} + X_{2T} \geq C(T)}\right] \\
=&S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T)} \int\int_{x_1 + x_2 \geq C(T)} e^{x_1} \phi_2(x|\mu, \Sigma) dx_1 dx_2 \\
=&S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T)} \int\int_{(1,1)x \geq C(T)} e^{(1,0)x} \phi_2(x|\mu, \Sigma) dx \\
=&S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T)} \exp\left((1,0)\mu + \frac{1}{2}(1,0)\Sigma(1,0)'\right) \Phi\left(\frac{(1,1)(\mu + \Sigma(1,0)') - C(T)}{\sqrt{(1,1)'\Sigma(1,1)}}\right) \\
=&S_0 e^{-\frac{1}{2}\sigma_S^2 T + B_1(T)} e^{\frac{1}{2}\Sigma_{11}^T} \Phi\left(\frac{\Sigma_{11}^T + \Sigma_{12}^T - C(T)}{\sqrt{\Sigma_{11}^T + 2\Sigma_{12}^T + \Sigma_{22}^T}}\right) \\
=&S_0 e^{B_1(T) + \frac{1}{2}(\Sigma_{11}^T - \sigma_S^2 T)} \Phi\left(\frac{\Sigma_{11}^T + \Sigma_{12}^T - C(T)}{\sqrt{D}}\right) \\
=&S_0 e^{B_1(T) + \frac{1}{2}(\Sigma_{11}^T - \sigma_S^2 T)} \Phi(d_1),
\end{aligned}$$

where $\phi_2(\cdot|a,b)$ is the probability density function of the two-dimensional normal distribution with mean vector a and covariance matrix b, $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution with $x=(x_1,x_2)'$, $\mu=(0,0)'$,

$$\Sigma=\begin{pmatrix}\Sigma_{11}^T & \Sigma_{12}^T\\ \Sigma_{21}^T & \Sigma_{22}^T\end{pmatrix},$$

$$D=\Sigma_{11}^T+2\Sigma_{12}^T+\Sigma_{22}^T,$$

and

$$d_1=\frac{\Sigma_{11}^T+\Sigma_{12}^T-C(T)}{\sqrt{D}}.$$

From the forth line to the fifth line, we used Kunitomo and Takahashi (1992) (See Lemma 1 in Kim 2002).

$$\begin{aligned}
(\mathrm{II}) =&Ke^{-B(T)}E^Q\left[e^{-X_{2T}}1_{X_{1T}+X_{2T}\geq C(T)}\right]\\
=&Ke^{-B(T)}\int\int_{(1,1)x\geq C(T)}e^{(0,-1)x}\phi_2(x|\mu,\Sigma)dx\\
=&Ke^{-B(T)}\exp\left((0,-1)\mu+\frac{1}{2}(0,-1)\Sigma(0,-1)'\right)\Phi\left(\frac{(1,1)(\mu+\Sigma(0,-1)')-C(T)}{\sqrt{(1,1)'\Sigma(1,1)}}\right)\\
=&Ke^{-B(T)}e^{\frac{1}{2}\Sigma_{22}^T}\Phi\left(\frac{-\Sigma_{12}^T-\Sigma_{22}^T-C(T)}{\sqrt{D}}\right)\\
=&Ke^{-B(T)+\frac{1}{2}\Sigma_{22}^T}\Phi(d_2)\\
=&KP^c(0,T)\Phi(d_2),
\end{aligned}$$

where

$$d_2=d_1-\sqrt{D}=\frac{-\Sigma_{12}^T-\Sigma_{22}^T-C(T)}{\sqrt{D}}.$$

C Implicit Eular-Maruyama Scheme

Following Alfonsi (2013), we discretize the CIR-type processes (3.6) in Monte-Carlo simulation. If $Y_t=\sqrt{r_t}$ ($Y_0=\sqrt{r_0}$), from Ito's formula, Y_t follows

$$dY_t=\left(\frac{\kappa\bar{r}-\sigma^2/4}{2Y_t}-\frac{\kappa}{2}Y_t\right)dt+\frac{\sigma}{2}dW_{1t}.$$

At this point, we consider the drift-implicit Euler-Maruyama approximation,

$$\hat{Y}_t=\hat{Y}_{t_k}+\left(\frac{\kappa\bar{r}-\sigma^2/4}{2\hat{Y}_t}-\frac{\kappa}{2}\hat{Y}_t\right)(t-t_k)+\frac{\sigma}{2}(W_{1t}-W_{1t_k})\tag{C.1}$$

for $t\in(t_k,t_{k+1}]$ with $t_k=\frac{kT}{n}$ $(T>0)$, and $\hat{Y}_0=\sqrt{r_0}$. Then, (C.1) gives the unique solution

$$\hat{Y}_t=\frac{\hat{Y}_{t_k}+\frac{\sigma}{2}(W_t-W_{t_k})+\sqrt{M(t_k,t)}}{2(1+\frac{\kappa}{2}(t-t_k))},\tag{C.2}$$

where $M(t_k, t) = (\hat{Y}_{t_k} + \frac{\sigma}{2}(W_t - W_{t_k}))^2 + 2(1 + \frac{\kappa}{2}(t - t_k))(a - \frac{\sigma^2}{4})(t - t_k)$. From Alfonsi (2013), if $\sigma^2 < \kappa\bar{r}$ and $1 \leq l < \frac{4}{3}\frac{\kappa\bar{r}}{\sigma^2}$, there exists a positive constant K_l such that

$$\left(E\left[\max_{t\in[0,T]} |\hat{Y}_t - Y_t|^l \right] \right)^{1/l} \leq K_l \frac{T}{n}.$$