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The Impact of Non-cash Collateralization on the Over-the-counter Derivatives Markets*

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Abstract

In this study, we propose a microeconomics model to verify the effects of the non-cash collateralization on the liquidity of the over-the-counter (OTC) derivatives markets accepting both cash and non-cash assets. Liquidity in this study is measured as an equilibrium volume of the derivatives contract. The equilibrium volume is obtained by solving the utility maximization problem of a risk-averse collateral payer who wants to optimize her/his capital. The collateral payer's capital depends on the non-cash asset used as collateral. We consider both option and forward contracts as example. Our sensitivity analysis shows that the optimal combination ofcash and non-cash collaterals can maximize the liquidity of derivatives. Especially, for option contracts, the market requires both cash and non-cash collaterals for liquidity. Overall, the introduction of non-cash collateralization boosts the liquidity of derivatives contracts. We also show how the arrangements of collateralization can boost the liquidity of the OTC derivatives markets. Moreover, we demonstrate that the combination of cash and non-cash collaterals to maximize liquidity differs from that to maximize the participant's utility. This indicates that the optimal combination is not efficient in terms of Pareto criteria.

JEL Classification: G10, G12, G13
Keywords: OTC derivatives markets, counterparty risk, non-cash collateralization, demand-supply analysis

1 Introduction

In this study, we propose a microeconomics model to show how the derivatives market functions under counterparty risk and non-cash collateralization. To mitigate and hedge counterparty risks, several methods have been suggested and utilized, that is, the adjustment of the derivatives price with the xVA, collateralization, and transferring the over-the-counter (OTC) transaction to central counterparty (CCP) transaction (Gregory 2015). In this study, we focus on collateralization.

A collateral is usually posted from an investor with negative exposure to an agent with positive exposure at a margin call or on the Marked-to-Market (MtM) date in a financial contract. In derivatives and other financial contracts, non-cash assets are used as collateral. For example, the

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CME permits participants to post the US treasury bills (T-Bills, TFRNs, T-Notes, T-Bonds) as collateral[1]. Moreover, news reports have shown the Osaka Exchange struggling to increase its share of derivatives transactions by expanding the type of collateral asset[2]. Therefore, we deduce that the expansion of collateral type for derivatives contracts might influence the derivatives market.

1.1 Our Research

Takino (2018b) considered a market model where the collateral is posted with only a non-traded asset in an option contract, and theoretically showed that the market volumes are lower under non-cash colllateralization than under cash collateralization. Meanwhile, the major security exchanges and clearing houses in the world are regulated to accept various non-cash assets as collateral. Therefore, from a practical point of view, the finding of Takino (2018b) does not support practical action. Our study proposes a market model to verify the advantage of the regulation accepting non-cash collaterals from the perspective of liquidity. Liquidity in this study is measured by the equilibrium volume of derivatives contract.

In Takino (2018b), all participants are risk-averse, and the collateral payer posts a non-cash asset as collateral by sourcing it from a repo market. However, in our model, the collateral payer (we suppose a bank) is risk-averse, has sufficient bonds as asset, and optimizes the derivatives contract volume to maximize her/his expected utility for the capital. Also, the participant posts the bond held by her/him as collateral (i.e., non-cash collateralization). The cash and non-cash asset combination posted as collateral is assumed to be given. The collateral receiver (we suppose a dealer) is assumed to be a risk-neutral price maker reflecting real world. This implies that the value of the derivatives at any time is calculated in the risk-neutral manner. We also assume that the collateral receiver reduces the non-cash collateral amount by haircut, and pays interest on cash collateral received, with interest calculated at the so-called collateral rate. For example, consider an option contract and a forward contract on a bond held by the collateral payer to motivate her/him to enter derivatives contracts. The option contract relates to the case of non-negative value for its buyer, whereas the forward contract corresponds to the case of both positive and negative values. By solving the collateral payer's optimization problem, we derive the optimal claim volume (or position). This contract simultaneously provides an equilibrium claim volume at the price offered by the collateral receiver, since the price quoted by the collateral receiver is not related to the volume as long as she/he adopts risk-neutral pricing. We numerically obtain the equilibrium volume under a stochastic model. We then carry out sensitivity analyses of the equilibrium volumes of derivatives pertaining to the parameters of non-cash collateralization. That is, we observe how the equilibrium volume depends on the cash and non-cash collateral combination, haircut, and collateral rate. At this point, we cannot explicitly solve the combination of the cash and non-cash collaterals optimal to maximize the expected utility of the collateral payer because of the tautology in optimizing and pricing. We thus numerically calculate the maximized expected utility of the collateral payer and determine the cash and non-cash collateral combination optimal to maximize her/his utility. This will also reveal whether the maximum liquidity in the derivatives market maximizes the collateral payer's welfare or not.

Our results are as follows. We first consider a model with no default. Then, we demonstrate that the equilibrium volume of the claim is maximum when the collateral payer pledges either cash or a non-cash asset as collateral. Furthermore, the collateral payer's posting preference depends

[1]https://www.cmegroup.com/clearing/files/acceptable-collateral-futures-options-select-forwards.pdf
[2]Nikkei Asian Review, Exchanges Compete for Dominance in Derivatives, April 25, 2017 (https://asia.nikkei.com/Business/Banking-Finance/).

on the relation between the risk-free rate and the collateral rate. For instance, when the risk-free rate is higher than collateral rate (this is true in the interest rate market), the collateral payer chooses non-cash collateralization. Next, we examine the model where the participant can default. The effects of the collateral assets combination on the equilibrium volume differ by the derivatives products. As regards the forward case, the maximum equilibrium volume is achieved when only the non-cash asset is pledged as collateral, as demonstrated in the non-default case. However, for the option case, an optimal combination of the cash and non-cash assets can maximize the volume, which is larger than that under full cash and full non-cash collateralization. Therefore, under the default setting, we find that both cash and non-cash collaterals are needed to maximize the liquidity of derivatives. However, we also demonstrate that the cash and non-cash collateral combination optimal maximize the equilibrium volume is different from that to maximize the utility of the collateral payer. That is, market optimality does not necessarily provide the participant's optimality in the choice of collateral assets. Moreover, if the collateral receiver is risk-neutral, the change in collateral assets combination improves the collateral payer's utility without reducing the utility level of the collateral receiver. Therefore, the maximized market volume is inefficient in the Pareto optimality sense. Finally, we report the effects of collateral arrangement (i.e., the haircut and collateral rate) on liquidity. Our sensitivity analyses provide intuitive results common for the option and forward contracts. That is, the reduction in haircut or increase in collateral rate enhances liquidity since the reduction in haircut decreases the collateral amount for the collateral payer and increase in collateral amount gives the collateral payer an opportunity to earn more interest.

Our contributions are summarized as follows:

1. We verify the impacts of the scope of collateral on the OTC derivatives markets,

2. We show the optimal combination of cash and non-cash collaterals to maximize the OTC derivatives market, which supports the practice,

3. We demonste that maximization of liquidity is not efficient for the participant.

1.2 Previous Literatures

This work is related to three topics of previous studies: (1)the reform of OTC derivatives transaction post the 2008 financial crisis and its impacts, (2)the effect of collateralization on asset markets, and (3) derivatives pricing with collateralization.

After the 2008 financial crisis, studies have proposed reforms for the regulation of the OTC derivatives transactions and studied their effects. Duffie and Zhu (2011) and Duffie et al. (2015) considered the collateral demand after the OTC transaction was transformed to a CCP transaction. Furthermore, Bellia et al. (2018) and Fiedor (2018) investigated the incentive to use the CCP. Their results depended on the market participant or product characteristics. This study focuses on the kind of assets posted as collateral rather than amount of collateral. Our results also depend on the derivatives product.

Collateralization has been discussed in the context of avoiding moral hazard. In fact, Acharya and Bisin (2014) considered an OTC derivatives market model to analyze the default incentive or moral hazard of the derivatives seller. Biais et al. (2016) discussed how collateralization reduced the moral hazard in the derivatives market. Furthremore, Geanakoplos (1996) demonstrated that collateralization increases liquidity and improves the Pareto efficiency. Taddei (2007) also showed how collateralization recovered the Pareto efficiency. Loon and Zhong (2016) empirically analyzed how transparency in the CDS markets affected liquidity. Our equilibrium approach is based on

Acharya and Bisin (2014) model, where participants with mean-variance utility maximized their expected utility in terms of derivatives position. This study examines how the combination of cash and non-cash collaterals impacts the liquidity and efficiency of the derivatives markets. Recall that liquidity is measured as the equilibrium volume of a claim. This scheme is based on Lo et al. (2004), who analyzed how transaction cost s affect the trading volume of assets. We, furthermore show that maximized liquidity does not achieve efficiency.

This study is also closely related to Takino (2016, 2018a, 2018b), who constructed equilibrium models for the derivatives markets and analyzed how the collateral amount affected the price and volume of derivatives contracts. Takino (2016) showed that an increase in collateral amount decreases the volume of an option contract, but a forward contract is not influenced by the collateral amount. Takino (2018a) incorporated the counterparty risk constraint, and demonstrated how the collateral amount affected an option market through the risk constraint. Takino (2018b) considered a market model where only the non-cash asset is used as collateral in an OTC derivatives contract, as introduced above. Our model is different from these studies in two ways. First, we use a risk-neutral pricing formula in deriving the derivatives values (along with the MtM values) to reflect the practice. The second is to focus on the scope of collateral assets rather than amount of collateral. Few studies considering collateralization focus on the kind of collateral asset, whereas almost all clearing houses or exchanges allow the market participants to post non-cash assets as collateral.

Collateralization in the derivatives contracts drastically changed the pricing formula for derivatives products even in the context of the risk-neutral criterion. The risk-neutral pricing formulae for derivatives took into account collateralization as proposed by several studies (Johannes and Sundaresan 2007; Fujii et al. 2010; Fujii and Takahashi 2013; Lou 2013, 2015, 2017; Crépey 2015a, 2015b; Takino 2019). While non-cash collateralization is explicitly incorporated in Lou (2017) and Takino (2019), among thers, previous studies show that derivatives pricing with collateralization essentially shows how to choose the discount rate. The selection of discount rate depends on how the collateral receiver uses the posted collateral, or the collateral payer sources the collateral assets including cash. In this study, the derivatives are priced from the view-point of a risk-neutral collateral receiver who invests the received collateral in risk-free assets using the repo for non-cash assets pledged as collateral. Thus, as in the previous studies, we derive a pricing formula and show that the discount is given by the net return, which the difference between the risk-free rate and funding rate.

1.3 Motivation from Technical View

We next consider a simple market model. The derivatives product issued at time 0 matures at time T. The participant with positive exposure at time 0 pays the (positive) derivatives value to the counterparty and settles the sale at time T. If the derivatives product is a European option, the buyer pays the option fee at time 0 and receives the option payoff at maturity. As shown in Tsuchiya (2016), by dividing the time period $[0, T]$ into N periods, the repeated buying and selling under the contract at each time period $[t_n, t_{n+1}]$ $(n = 0, 1, \ldots, N-1)$ is equivalent to buying the derivatives at time 0 $(= t_0)$ and selling them at time T $(= T_N)$ (Figure 1). Therefore, we consider only the time period $[t_n, t_{n+1}]$, and set $\Delta t = t_{n+1} - t_n$ for all n.

Consider an investor who holds M units of bonds and enters a derivatives contract with a negative exposure simultaneously posting collateral with cash or bond or both. A participant with negative exposure enters the derivatives contract by receiving the derivatives value from the counterparty with a positive exposure. The cash thus received is used as cash collateral. The collateral receiver should pay interest (at the collateral rate) on the posted cash collateral whether

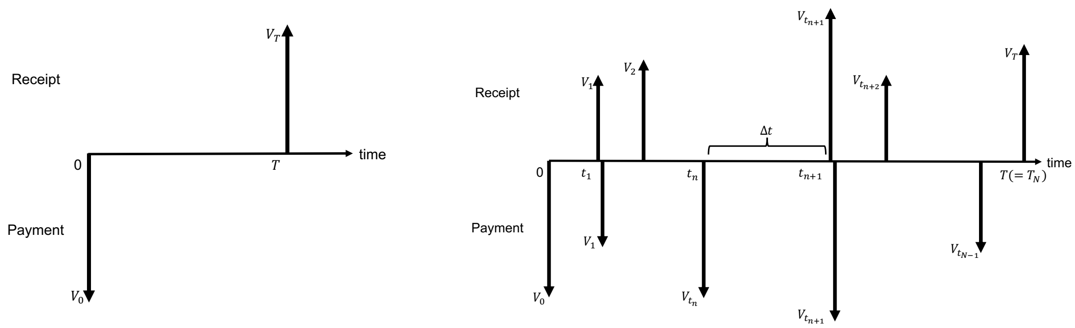

Figure 1: Partitioning the derivatives contract during $[0, T]$. Both figures show the cash flows that the buyside (or asset side) receives. The left-hand side shows the derivatives contract issued at time 0 and matured at time T. The right-hand side divides the time to maturity $[0, T]$ into N partitions. The figure shows that in each period $[t_n, t_{n+1}]$, the participant repeats to buy and sell per the contract. The figure is based on Figure 8-3 in Tsuchiya (2016).

a default occurs or not. We denote the collateral rate at time t by r_t^c. Haircut $h \times 100\%$ is applied to non-cash collateralization. For example, the collateral receiver evaluates the collateral value as $100\times(1-h)$ if the counterparty posts the non-cash collateral with a value of 100. Thus, the collateral payer should post a higher collateral when she/he pays the non-cash collateral. We denote the cash account and value of the bond held by the collateral payer at time t as A_t^C and A_t^B, respectively.

Recall that the participant with negative exposure at the contract date enters k units of the derivatives contract and receives cash amounts to the value of the derivatives from the counterparty who has a positive exposure. Now, the cash account and value of the bond held by the collateral payer at the contract date t_n are, respectively,

$$A_{t_n}^C = k_{t_n}|V_{t_n}| - \eta k_{t_n}|V_{t_n}| = (1-\eta)k_{t_n}|V_{t_n}|,$$

$$A_{t_n}^B = MB_{t_n} - k_{t_n}\frac{1-\eta}{1-h}\frac{|V_{t_n}|}{B_{t_n}}B_{t_n} = \left(M - k_{t_n}\frac{1-\eta}{1-h}\frac{|V_{t_n}|}{B_{t_n}}\right)B_{t_n},$$

where η is the proportion of cash collateral to the total collateral amount (we call it the cash collateral ratio), and automatically $(1-\eta)$ means the proportion of non-cash collateral, V_t is the derivatives value per unit at time t from the point of view of the collateral receiver (assuming $V_t > 0$), and B_t is the bond price at time t.

At the next MtM date (i.e., time t_{n+1}), in case of no default, the collateral payer returns the posted collateral, that is, $\eta \times 100\%$ of the cash collateral and $(1-\eta) \times 100\%$ of the non-cash collateral. The collateral payer also earns interest on the cash collateral. Now, the cash account and value of the bond held by the collateral payer at the contract date t_n are, respectively,

$$\begin{aligned}A_{t_{n+1}}^C =&(1 + r_{t_n}\Delta t)A_{t_n}^C + (1 + r_{t_n}^c\Delta t)\eta k_{t_n}|V_{t_n}| \\ =&\{(1-\eta)(1 + r_{t_n}\Delta t) + \eta(1 + r_{t_n}^c\Delta t)\}k_{t_n}|V_{t_n}| \\ =&k_{t_n}|V_{t_n}| + ((1-\eta)r_{t_n} + \eta r_{t_n}^c)k_{t_n}|V_{t_n}|\Delta t,\end{aligned} \tag{1.1}$$

$$\begin{aligned}A_{t_{n+1}}^B =&\left(M - k_{t_n}\frac{1-\eta}{1-h}\frac{|V_{t_n}|}{B_{t_n}}\right)B_{t_{n+1}} + k_{t_n}\frac{1-\eta}{1-h}\frac{|V_{t_n}|}{B_{t_n}}B_{t_{n+1}} \\ =&MB_{t_{n+1}}.\end{aligned}$$

Therefore, by posting cash as a collateral, the collateral payer increases her/his asset amount by

$$\Delta A^C := A^C_{t_{n+1}} - A^C_{t_n} = \eta k_{t_n}|V_{t_n}| + ((1-\eta)r_{t_n} + \eta r^c_{t_n})k_{t_n}|V_{t_n}|\Delta t (> 0),$$

and the volume of the bond held by the collateral payer remains unchanged in case of no default.

Now, since

$$\frac{\partial \Delta A^C}{\partial \eta} = k_{t_n}|V_{t_n}| + (r^c_{t_n} - r_{t_n})k_{t_n}|V_{t_n}|\Delta t, \tag{1.2}$$

from (1.1), we interpret the relationship (1.2) as follows. The collateral payer is willing to post the cash collateral ($\eta \nearrow 1$) if $r^c > r$. However, when $r > r^c$, the collateral payer is willing to post the non-cash asset collateral ($\eta \searrow 0$). This means that the collateral payer must choose either a cash or non-cash asset as collateral.

Note that the above intuition assumes no default. If the collateral payer actually defaults, then she/he will lose

$$\eta k_{t_n}|V_{t_n}| + k_{t_n}\frac{1-\eta}{1-h}\frac{|V_{t_n}|}{B_{t_n}}B_{t_{n+1}}.$$

Therefore, we deduce that the collateral payer finds collateral asset selection is more complicated when a default occurs. Moreover, when the participant is risk-averse, she/he might avoid choosing only either cash or non-cash asset as collateral. In this study, we show how the collateral payer combines the collateral assets, and then verify the validity of the recent clearing house actions.

The rest of the paper is organized as follows. The next section sets up the model. We then define the market participants in our derivatives market, collateralization, and the behaviors of the participants. Section 3 provides a formula and an equilibrium volume for the derivatives by solving the optimization problem of the collateral payer. Section 4 numerically carries out the sensitivity analyses. This shows how non-cash collateralization affects the liquidity of the derivatives contract. Section 5 concludes our study.

2 Model and Collateralization

2.1 Financial Market

The derivatives market has two market participants, the dealer and the bank. They enter into a kind of derivatives contract, and do not manage derivatives assets as a portfolio. We assume that the dealer is risk-neutral and a price-maker, and the bank is risk-averse and price-taker, for all derivatives products. We denote the value of the derivatives contract from the view-point of the dealer at time t as V_t. We further assume that the dealer has a positive exposure in the derivatives contract (i.e., $V_t > 0$ at the contract date), and that the bank has a negative exposure at the contract date. Also, the bank has to post a collateral to the dealer.

We next assume that our financial market is trading a zero-coupon bond with the highest credit rating and maturity T_∞. Also, the bond has a considerably longer maturity than any other derivatives maturity; the time-t price of the bond with maturity T_∞ is denoted by $B_t = B(t, T_\infty)$. The bank has a number of bonds as main asset, and uses them as non-cash collateral asset. We also suppose that the derivatives are written on the bond, and that this assumption motivates the bank to trade the derivatives.

2.2 Collateral Agreement

A participant with negative exposure pledges collateral to the counterparty with positive exposure at each MtM date; that is, the participants pay a variation margin during the life of the derivatives contract. We assume that no initial margin is applied. The MtM is continuous, as defined in Fujii and Takahashi (2013). The collateral amount per derivatives contract is equal to the value of the derivative (perfect collateralization). Cash and non-cash assets are accepted as collateral. We assume that the proportion of cash to total collateral is exogenously given. We also assume that haircut h ($0 \leq h \leq 1$) is applicable for non-cash collateralization. Thus, the collateral payer has to post a larger amount of non-cash asset when using it as collateral.

The collateral receiver has to pay the collateral rate r^c on the cash collateral and return; the non-cash collateral bears this rate. Now, the collateral receiver actually pays no return for the non-cash collateral since the collateral payer uses a zero-coupon bond as non-cash collateral. The posted collateral is returned if the counterparty does not default; no collateral is returned otherwise. We assume zero recovery, that is, the investor with positive exposure receives no payment at default.

2.3 Behaviors of Market Participants

The dealer is the price maker in the derivatives market, and is risk-neutral. Hence, the derivatives' price is determined in a risk-neutral manner with collateralization. The bank is a price taker in the derivatives market, and is risk-averse. We assume that the risk preference is presented as mean-variance utility; this provides us with the explicit formula of the equilibrium volume for the derivatives. The bank determines the contract volume k of the derivatives to maximize their expected utility for capital L at the next MtM date (Danielsson et al. 2009).

3 Analysis

3.1 Pricing by Dealer

In this section, we provide a pricing formula for the derivatives; it can be used to also evaluate the MtM value of the derivatives.

The dealer has a positive exposure in the derivatives contract at the contract date; that is, she/he receives collateral from the counterparty on the same day. The cash part of the received collateral is invested in a risk-free asset at the risk-free rate r, and the non-cash part is exchanged for cash through the repo market with haircut h_p ($0 \leq h_p \leq 1$); the cash thus obtained is again invested at the risk-free rate r. In order to ensure perfect collateralization for the collateral receiver, we assume that

$$h \equiv h_p.$$

The funds sourced via the repo market is returned to the repo market at the repo rate r^p, to receive the asset. This asset is returned to the counterparty if she/he does not default.

Proposition 3.1. *Let r be a risk-free rate, r^c be the collateral rate, and r^p be the repo rate. For any cash collateral ratio η ($0 \leq \eta \leq 1$), the risk-neutral price of the (continuously, perfectly) collateralized derivatives is given by*

$$V_t = E_t^Q \left[e^{-\int_t^T (r_s - y_s) ds} V_T \right], \tag{3.1}$$

where E_t^Q is the expectation under the risk-neutral measure Q conditioned with no default up to t, and

$$y_s = r_s - (\eta r_s^c + (1-\eta) r_s^p).$$

Proof. The dealer deposits the posted cash collateral ηV_t with risk-free rate r and the returns with collateral rate r^c. The dealer also sources cash $\frac{1-h_p}{1-h}(1-\eta)V_t = (1-\eta)V_t$ (under the assumption $h = h_p$) by exchanging it for the posted bond worth $\frac{1}{1-h}(1-\eta)V_t$ in the repo market. The money sourced via the repo market is further deposited with risk-free rate r and the returns are deposited with the repo rate r^p. Thus, the instantaneous change in collateral for the collateral receiver is

$$y_t V_t dt := \{r_s - (\eta r_s^c + (1-\eta) r_s^p)\} V_t dt.$$

Therefore, the derivatives' time t value is given by

$$\begin{aligned} V_t =& E_t^Q \left[\left\{ e^{-\int_t^T r_s ds} V_T + \int_t^T e^{-\int_t^s r_u du} y_s C_s ds \right\} 1_{\tau > T} \right] \\ &+ E_t^Q \left[\left\{ e^{-\int_t^\tau r_s ds} C_\tau + \int_t^\tau e^{-\int_t^s r_u du} y_s C_s ds \right\} 1_{\tau \le T} \right], \end{aligned} \tag{3.2}$$

where τ is the default time. The first expectation shows that the derivatives payment included the net return from investing the posted collateral without defaults, and the second expectation shows that the collateral value included the net return from investing the posted collateral at default. (3.2) agrees with (A.1) of Johaness and Sundaresan (2007). The assumption of continuous and perfect collateralization yields

$$C_t = V_t \tag{3.3}$$

for $0 \le t \le T$. From Johaness and Sundaresan (2007), (3.2) reduces to

$$V_t = E_t^Q \left[e^{-\int_t^T r_s ds} V_T + \int_t^T e^{-\int_t^s r_u du} y_s V_s ds \right]. \tag{3.4}$$

Thus, the derivatives pricing formula under full collateralization becomes

$$V_t = E_t^Q \left[e^{-\int_t^T (r_s - y_s) ds} V_T \right].$$

□

Remark 3.1. *1. At the default time τ, the participant with positive exposure recovers the loss due to default from the collateral including non-cash asset. However, the non-cash asset can vary over time even when it is a highly rated government bond. Thus, (3.3) does not necessary hold at the settlement date of the collateral after default. Of course, $C_\tau < V_\tau$ is crucial. In practice, the initial margin has to cover this deterioration. In our study, we ignore this risk by assuming continuous and perfect collateralization without loss of generality.*

2. (3.4) shows the terminal value of the derivatives contract, including the net return of collateral without the default obtained in Johaness and Sundaresan (2007), Fujii et al. (2010), Fujii and Takahashi (2016), and Takino (2019). Recall that this characteristic arises from the assumption of perfect and continuous collateralization. Therefore, we can use the pricing formula (3.1) under both defaultable and non-defaultable situations.

B/S	
Bond (MB_{t_n-})	Debt (D_{t_n-})
	Capital (L_{t_n-})

Table 1: Bank balance sheet before entering the derivatives contract. t_n- means just before contracting the derivatives at time t_n.

3.2 Bank's Problem

Next, we consider the bank's optimization problem for the time period $[t_n, t_{n+1}]$ as introduced in Section 1.3. We assume that the bank initially holds M units of bonds (M is given) with debt amount D_{t_n-} just before the contract date t_n (Table 1). The participants enter k_{t_n} units of the derivatives contract. We assume $V_{t_n} > 0$ to highlight the collateral posting. That is, the bank has a negative exposure and should post the collateral with $\eta \times 100\%$ of the cash collateral and $(1-\eta) \times 100\%$ of the non-cash collateral to the MtM value of the derivatives at t_n. The dealer has a positive exposure and pays V_{t_n} to the bank as derivatives fee. We denote the capital of the bank at time t_n by L_{t_n}; that is,

$$L_{t_n} = MB_{t_n} - D_{t_n} + k_{t_n}|V_{t_n}| - (1-\eta)k_{t_n}m_{t_n}B_{t_n} - \eta k_{t_n}|V_{t_n}|, \tag{3.5}$$

where

$$m_t = \frac{1}{1-h}\frac{|V_t|}{B_t}$$

and $k_{t_n}|V_{t_n}|$ is the derivatives contract value paid by the dealer. The fourth and fifth terms are the cash and non-cash collateral postings, respectively.

At the next MtM date t_{n+1}, the bank recovers the posted collateral and settles the derivatives position if it does not default (Figure 2), with no settlement otherwise, except to receive the interest for the posted cash collateral (Figure 3). For simplicity, we assume that the default payment is settled at t_{n+1} even if the default occurs before t_{n+1}. Then, the capital of the bank $L_{t_{n+1}}$ at the next MtM date, that is, t_{n+1} of the derivatives, is

$$\begin{aligned} L_{t_{n+1}} =&(M - k_{t_n}(1-\eta)m_{t_n})B_{t_{n+1}} - D_{t_{n+1}} - k_{t_n}(V_{t_{n+1}} - (1-\eta)m_{t_n}B_{t_{n+1}} - \eta|V_{t_n}|)1_{\tau > t_{n+1}} \\ &+ r^c_{t_n}k_{t_n}\eta|V_{t_n}|\Delta t + k_{t_n}(1+r_{t_n}\Delta t)(1-\eta)|V_{t_n}| \\ =&MB_{t_{n+1}} - D_{t_{n+1}} + k_{t_n}(1+r_{t_n}\Delta t)(1-\eta)|V_{t_n}| + k_{t_n}r^c_{t_n}\eta|V_{t_n}|\Delta t - k_{t_n}g(t_{n+1};\eta), \end{aligned}$$

where

$$g(t_n;\eta) = g_0(t_n;\eta)1_{\tau > t_n} + g_1(t_n;\eta)1_{\tau \le t_n}, \tag{3.6}$$

and

$$\begin{aligned} g_0(t_n;\eta) &= V_{t_n} - \eta|V_{t_{n-1}}|, \\ g_1(t_n;\eta) &= (1-\eta)m_{t_{n-1}}B_{t_n}. \end{aligned}$$

Recall that the bank has a mean-variance utility; that is, her/his utility function is represented as

$$U(X) = E[X] - \frac{\gamma}{2}Var[X]$$

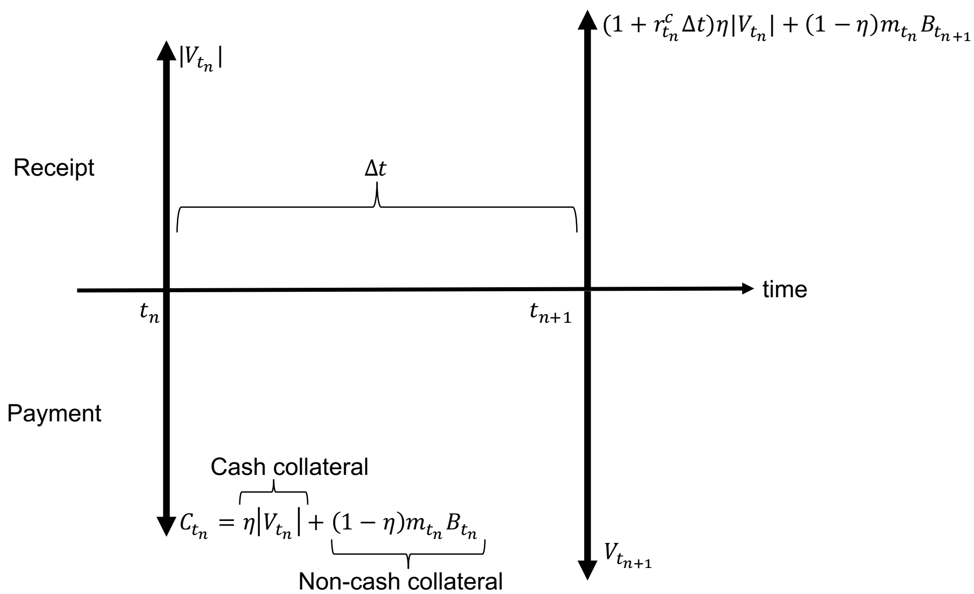

Figure 2: Cash flows of the bank for a unit of derivatives contract in case of no default. C_{t_n} is the collateral amount.

for random wealth X, where γ is a risk-aversion coefficient. Then, the utility maximization problem of the bank is represented by

$$\max_k E_{t_n}[U(L_{t_{n+1}})] = \max_k \left[E_{t_n}[L_{t_{n+1}}] - \frac{\gamma}{2}Var_{t_n}[L_{t_{n+1}}]\right], \tag{3.7}$$

where E_t and Var_t are respectively the expectation and variance conditioned with no default up to time t.

The optimization problem (3.7) is solved statically. By the first-order condition, the solution of (3.7) is

$$k^*_{t_n}(V_{t_n}) = \frac{-E_{t_n}[g(t_{n+1};\eta)] + (1 + r_{t_n}\Delta t)(1-\eta)|V_{t_n}| + r^c_{t_n}\eta|V_{t_n}|\Delta t + \gamma M Cov_{t_n}[B_{t_{n+1}}, g(t_{n+1};\eta)]}{\gamma Var_t[g(t_{n+1};\eta)]}, \tag{3.8}$$

where Cov_t is a covariance operator conditioned with no default up to t. This implies the demand or supply function for the derivatives.

3.3 Equilibrium

Recall that the derivatives price is determined by the dealer who is the price maker. Thus, the dealer's price is also the equilibrium price. However, the equilibrium volume is obtained by substituting the equilibrium price (3.1) into (3.8). We solve the equilibrium price and volume numerically after setting a stochastic model for the bond price and interest rates.

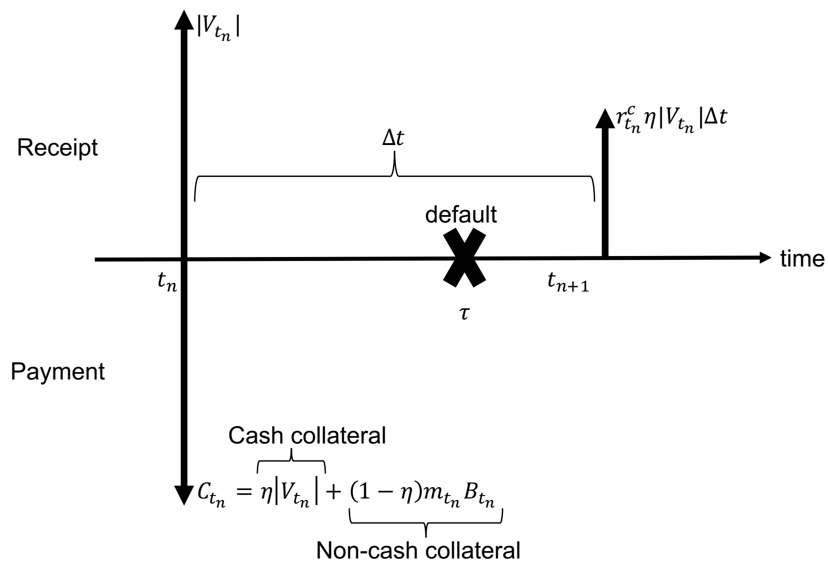

Figure 3: Cash flows of the bank for a unit of derivatives contract in case of default. C_{t_n} is the collateral amount.

4 Numerical Result

4.1 Model

Assume a filtered (physical) probability space $(\Omega, P, \mathcal{F}, \mathcal{F}_t)$, and let $W_t = (W_{1t}, W_{2t}, \ldots, W_{4t})$ be a four-dimensional standard Brownian motion and the filtration $\mathcal{F}_t^W$ be generated by the Brownian motion,

$$\mathcal{F}_t^W = \sigma(W_s; s \leq t).$$

Next, let each asset price be assumed to be driven by the following stochastic differential equations; the bond price process is

$$\frac{dB_t}{B_t} = \mu_B dt + \sigma_B dW_{1t},$$

the risk-free rate is

$$dr_t = \kappa_r(a_r - r_t)dt + b_r dW_{2t},$$

the collateral rate is

$$dr_t^c = \kappa_c(a_c - r_t^c)dt + b_c dW_{3t},$$

and the repo rate is

$$dr_t^p = \kappa_p(a_p - r_t^p)dt + b_p dW_{4t},$$

where μ_B, σ_B, $\kappa_\cdot$, $a_\cdot$, and $b_\cdot$ are all constant. As regards the interest rate models, we use the Vasicek type for convenience.

Next, we model the default event of the bank according to Schönbucher (2003). We use the so-called reduced-form model, and define the default time τ as the first jump time of Poisson process G. That is,

$$\tau = \inf\{t > 0 | G_t > 0\}.$$

In our study, the default event is explicitly included in the bank's optimization problem for a very shot time period $[t_n, t_{n+1}]$, rather than in the dealer's pricing. As guided by Fujii and Takahashi (2013), the margin call is done daily. We also assume that the time period Δt is one day. Hence, we have the following assumption without loss of generality.

Assumption 4.1. *We assume that all interest rates and the intensity of the Poisson process G are constant for $[t_n, t_{n+1}]$ in the bank's optimization problem.*

From Assumption 4.1, we set the intensity of the Poisson process as $\lambda_t \equiv \lambda$ for $t_n \le t \le t_{n+1}$ in the following numerical implementation, and treat all interest rates as constant during $[t_n, t_{n+1}]$; that is, $r_t \equiv r$, $r_t^c \equiv r^c$, and $r_t^p \equiv r^p$ for $t_n \le t \le t_{n+1}$.

Now, we define the filtration $\mathcal{F}_t^G$ generated by the Poisson process N as

$$\mathcal{F}_t^G = \sigma(G_s; s \le t),$$

and set

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^G.$$

Finally, for the pricing derivatives, we introduce the equivalent martingale measure Q as

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left\{-\int_t^T \theta(s) dW_{1s} - \frac{1}{2}\int_t^T \theta(s)^2 ds\right\},$$

where

$$\theta(t) = \frac{\mu_B - r_t}{\sigma_B}.$$

Under measure Q, each stochastic process is represented as

$$\frac{dB_t}{B_t} = r_t dt + \sigma_B d\tilde{W}_{1t},$$

$$\begin{aligned} dr_t &= \kappa_r(a_r - r_t)dt + b_r d\tilde{W}_{2t}, \\ dr_t^c &= \kappa_c(a_c - r_t^c)dt + b_c d\tilde{W}_{3t}, \\ dr_t^p &= \kappa_p(a_p - r_t^p)dt + b_p d\tilde{W}_{4t}, \end{aligned}$$

where $\tilde{W} = (\tilde{W}_1, \tilde{W}_2, \ldots, \tilde{W}_4)$ is the four-dimensional Brownian motion under measure Q. Under these stochastic models, we must obtain the closed-form pricing formulae for some derivatives products as demonstrated in Takino (2019). We use the formulae derived by Takino (2019) for numerical implementation.

The fundamental parameter values used in this simulation are as follows: $B_0 = 92.0$, $B_{t_n} = 95.0$, $\mu_B = 0.1$, $\sigma_B = 0.2$, $r_0 = r_{t_n} = 0.03$, $\kappa_r = 1.0$, $a_r = 0.025$, $b_r = 0.03$, $\kappa_c = 1.0$, $a_c = 0.025$, $b_c = 0.02$, $r_0^p = r_{t_n}^p = 0.025$, $\kappa_p = 1.0$, $a_p = 0.02$, $b_p = 0.03$, $\lambda_0 = 0.02$, $\gamma = 0.02$, and $M = 10,000,000$. The frequency of Monte-Carlo simulation is 1,000,000 times, and we divide one year into 1,000 time grids (i.e., $\Delta t = \frac{1}{1000}$). The remaining parameters are defined in each section. However, we assume that

$$r_0 > r_0^c, \quad r_{t_n} > r_{t_n}^c$$

without loss of generality.

Now, if the derivatives contract is entered into at time t_n, the derivatives position will be cleared at time t_{n+1} $(= t_n + \Delta t)$. The derivative values are priced at both t_n and t_{n+1}. We assume that the time to maturities from t_n and t_{n+1} are approximately equal; that is,

$$T - t_n \approx T - t_{n+1},$$

because Δt is very small.

4.2 The Benchmark Results: Non-default Case

We first consider a situation where the bank never defaults. The non-default setting is achieved by applying $\lambda = 0$. Then, we have

$$g(t;\eta) = g_0(t;\eta)$$

for $0 \le t \le T$. This means that the derivatives payoff with collateralization is independent of the haircut h. Moreover, the equilibrium volume (3.8) is represented as

$$k_{t_n}^*(V_{t_n}) = \frac{-E_{t_n}[V_{t_{n+1}}] + |V_{t_n}| + r_{t_n}|V_{t_n}|\Delta t - \eta(r_{t_n} - r_{t_n}^c)|V_{t_n}|\Delta t + \gamma MCov_{t_n}[B_{t_{n+1}}, V_{t_{n+1}}]}{\gamma Var_t[V_{t_{n+1}}]}. \tag{4.1}$$

This does not depend on the haircut h. For (4.1), we have

$$\frac{\partial k_{t_n}^*(V_{t_n})}{\partial \eta} = -\frac{(r_{t_n} - r_{t_n}^c)|V_{t_n}|\Delta t}{\gamma Var_t[V_{t_{n+1}}]}. \tag{4.2}$$

Then, when $r_{t_n} > r_{t_n}^c$, we have

$$\frac{\partial k_{t_n}^*(V_{t_n})}{\partial \eta} < 0.$$

Thus, the cash collateral ratio η decreases the contract volume. In other words, the market prefers non-cash collateralization if no participant defaults. The bank is then willing to invest in the risk-free asset, rather than post cash as collateral, because they can earn more money when $r_{t_n} > r_{t_n}^c$. However, from (4.2), the bank wants to post cash as collateral if $r_{t_n}^c > r_{t_n}$. These results are for the non-default case. In the following sections, we observe how the optimal combination of cash and non-cash collaterals changes under default.

Finally, from (4.1), we have

$$\frac{\partial k_{t_n}^*(V_{t_n})}{\partial r_{t_n}^c} = \frac{\eta|V_{t_n}|\Delta t}{\gamma Var_t[V_{t_{n+1}}]} > 0.$$

Thus, an increase in collateral rate raises the equilibrium volume of the derivatives. The collateral rate is a return for the collateral payer, who certainly earn more interest by posting the cash collateral when the collateral rate increases. This makes the collateral payer more willing to enter the derivatives contract.

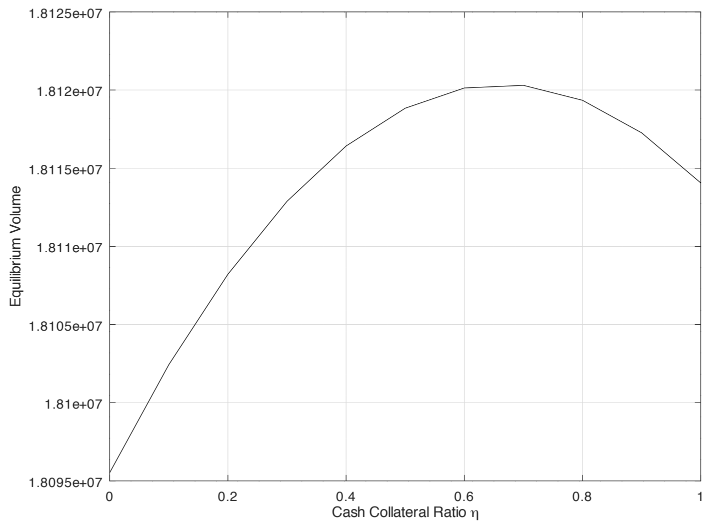

Figure 4: Cash collateral ratio η and the equilibrium volume of option contract

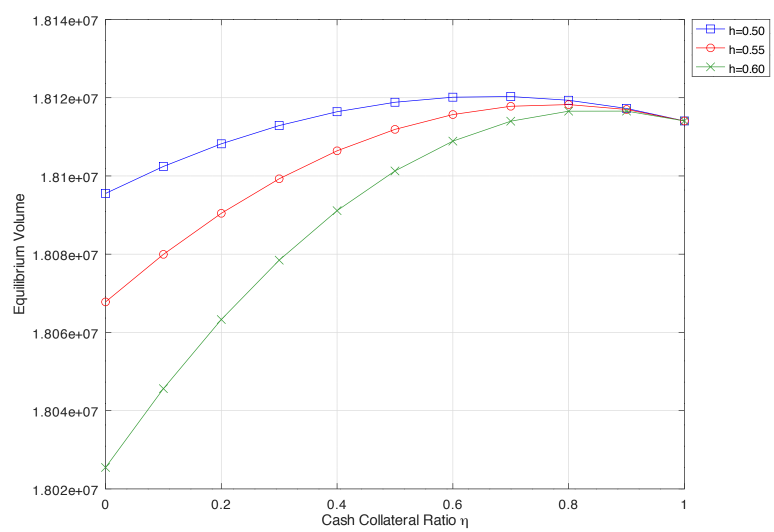

Figure 5: Cash collateral ratio η and the equilibrium volume of the option contract for each haircut h. Simulations are implemented for $h = 0.1, 0.2$, and 0.3.

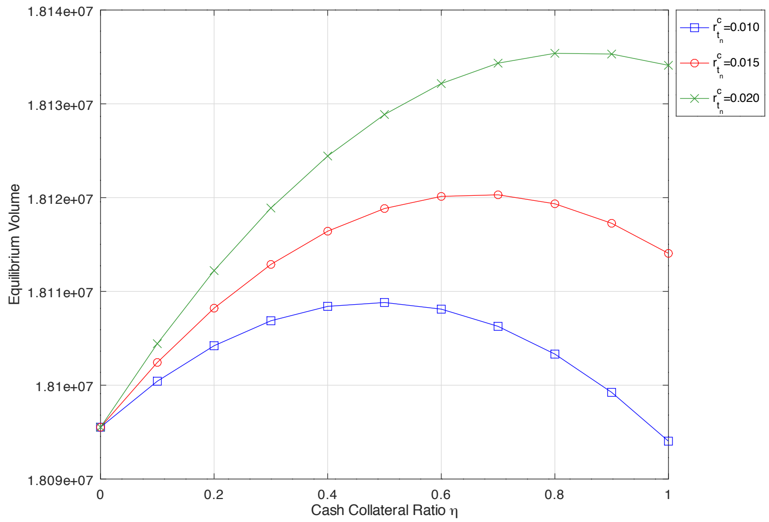

Figure 6: Cash collateral ratio η and the equilibrium volume of the swap contract for each $r^c_{t_n}$. Simulations are implemented for $r^c_0 = 0.010, 0.015$, and 0.020.

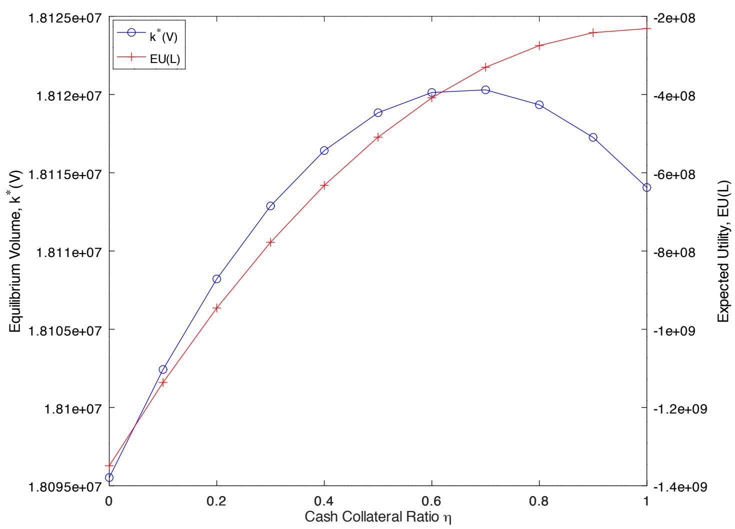

Figure 7: Cash collateral ratio η with the equilibrium volume, and bank's expected utility of the option contract

4.3 Example: A Bond Option

We consider the case of a call option written on the bond. We set the maturity of the option as $T = 0.25$ ($\ll T_\infty$) and the payoff function as

$$V_T = \max(B_T - K_B, 0),$$

where $B_T = B(T, T_\infty)$ and K_B is a strike price; we then set $K_B = 95.0$. We assume that the option is issued at time 0 ($= t_0$) and the option contract is entered at time 0. The dealer purchases the option at time 0 from the bank and sells it at time t_1. Since the dealer is an option buyer, the option price is larger than zero (i.e. $V_0 > 0$) as assumed. Then, the dealer pays V_0 at the contract date 0, and receives V_{t_1} from the bank. From Proposition 3.1, the option fee V_0 that the dealer pays to the bank is

$$V_0 = E^Q[e^{-\int_0^T (r_s - y_s)ds} V_T].$$

The equilibrium volume is obtained by plugging V_0 into (3.8).

First, Figure 4 shows the relation between the cash collateral ratio η and the equilibrium volume at $h = 0.50$ and $r_0^c = 0.015$. The figure also shows a non-linear relationship (i.e. a convex) between the cash collateral ratio and equilibrium volume, as well as the optimal cash collateral ratio to maximize the volume. The maximum volume is achieved at $\eta = 0.70$. That is, we find the combination of cash and non-cash collaterals optimal to maximize the contract volume; this volume is larger than that at full cash and non-cash collateralization. This result is straightforward, because the collateral payer is a risk-averse investor with a mean-variance utility. However, this result contradicts those obtained for cases where no default occurs (Section 4.2). Thus, in cases where the claim is defaultable and the participant is risk-averse, the market prefers to combine the cash and non-cash collaterals, rather than accept either asset only.

Next, we examine how the haircut h applied to non-cash collateral influences the option market. Figure 5 shows the relationship between the cash collateral ratio η and equilibrium volume for each haircut h. We consider $h = 0.50, 0.55$, and 0.60 for $r_0^c = 0.015$. In the figure, the line for $h = 0.50$ lies at the top and that for $h = 0.60$ lies at the bottom. That is, the haircut reduces the volume. When the haircut is high, the collateral payer should post more non-cash collateral based on the amount discounted with the haircut. Then, on default, the participant loses more of the asset posted as collateral. This prevents the participant from entering the derivatives contract. This explains why the volume declines when the haircut increases. Furthermore, the figure shows that the optimal cash collateral ratio maximizing the equilibrium volume varies over the haircut. The optimal cash collateral ratio is 70%, 80%, and 90% at $h = 0.50$, $h = 0.55$, and $h = 0.60$, respectively. That is, the increase in haircut raises the optimal cash collateral ratio for liquidity. Recall that the collateral payer loses more of her/his own asset with higher haircut on default. This prevents the participant from posting the collateral with non-cash asset.

Next, we investigate how the collateral rate influences the option market. We examine a simulation for each $r_0^c = 0.010, 0.015$, and 0.020 with $h = 0.50$. Figure 6 shows the relationship between the cash collateral ratio η and equilibrium volume for each initial collateral rate r_0^c. From the figure, the line of $r_0^c = 0.020$ lies at the top and that of $r_0^c = 0.010$ lies at the bottom. That is, the collateral rate boosts the equilibrium volume. An increase in collateral rate yields more returns for the collateral payer, and she/he is willing to enter into more contracts even if the amount of the posted collateral increases due to the increase in position. This explains why the increase in collateral rate raises the contract volume. Moreover, from the figure, the cash collateral ratio optimal for maximizing the volume depends on the collateral rate. The optimal cash collateral ratio is 50%, 70%, and 80% at $r_0^c = 0.010$, $r_0^c = 0.015$, and $r_0^c = 0.020$, respectively. That is, the collateral rate increases the optimal cash collateral ratio for liquidity, or the market prefers the cash collateral to be posted when the collateral rate increases. The implication of the result is straightforward. The increase in collateral rate brings the collateral payer higher interest for the posted cash collateral even if she/he defaults. The collateral payer tends to post more cash collateral when the collateral rate increases.

Finally, we implement the expected utility $E_{t_n}[U(L_{t_n+1})]$ of the bank under $h = 0.50$ and $r_0^c = 0.015$. While we have obtained the optimal cash collateral ratio to maximize the market volume of the option contract, we have not solved the utility maximization problem of the bank with respect to the cash collateral ratio. Intuitively, one can find the optimal cash collateral ratio to maximize the expected utility of the bank by solving the bank's utility maximization problem. However, the change in cash collateral ratio simultaneously changes the claim price. This makes it complicated to analytically solve the problem. Hence, we rely on the numerical method. In this examination, we confirm the consistency between the preferences of the market and the collateral payer. Figure 7 shows the graphs of the equilibrium volume (marked as "$\circ$"; the corresponding axis is on the left-hand side) and the expected utility (marked as "+"; the corresponding axis is on the right-hand side) respectively. We observe that the maximum equilibrium volume is at $\eta = 0.7$ and the maximum expected utility is at $\eta = 1.0$. That is, the cash collateral ratio η maximizing the equilibrium volume differs from that maximizing the expected utility. This means that the bank (or collateral payer) prefers to post collateral by cash only, whereas the market prefers to include both the cash and non-cash collaterals. At this point, the dealer is risk-neutral. This enables us to assume that she/he has a linear (expected) utility function of the wealth under the risk-neutral measure Q. This also shows that the expected utility of the dealer will vanish if the derivatives price is calculated by the risk-neutral pricing rule. Thus, the utility of the dealer does not depend on the derivatives. Recall that Figure 7 shows that the expected utility of the bank increases when one increases the cash collateral ratio from the optimal level (i.e., $\eta = 0.70$). Because the dealer does not depend on the derivatives, one can increase the utility of a participant without reducing another one. Hence, the combination of cash and non-cash collaterals optimal maximize the liquidity is not efficient by means of Pareto criteria.

4.4 Example: A Forward Contract on Bond

Consider a forward contract on bond. Assume that the forward contract has been issued at time 0 ($= t_0$), and that the bank enters into the contract with the dealer at time t_n and settles the position at the next MtM date (at time t_{n+1}). We set $r_0^c = 0.015$ for this example. We also assume that $V_{t_n} > 0$ (i.e., the dealer has positive exposure at the contract date). Then, the dealer pays V_{t_n} to the bank at the contract date t_n and receives $V_{t_{n+1}}$ from the bank if $V_{t_{n+1}} > 0$, and pays $|V_{t_{n+1}}|$ to the bank otherwise. Denoting the forward price by F_B determined at time 0, the value of the forward contact from the dealer's perspective at time $t > 0$ is given by

$$V_t = E_t^Q \left[e^{-\int_t^T (r_s - y_s) ds} (B_T - F_B) \right], \tag{4.3}$$

where F_B is given by

$$F_B = \frac{E^Q \left[e^{-\int_0^T (r_s - y_s) ds} B_T \right]}{E^Q \left[e^{-\int_0^T (r_s - y_s) ds} \right]}.$$

When this is solved, the value of the forward contract at time 0 is equal to 0; that is, $V_0 = 0$. The equilibrium volume of the forward contract is obtained by substituting V_{t_n} in (4.3) into (3.8). As in

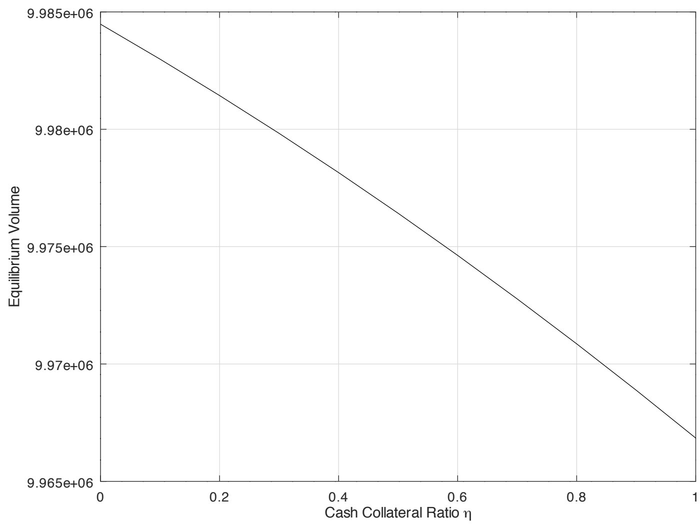

Figure 8: Cash collateral ratio η and market size of forward contract

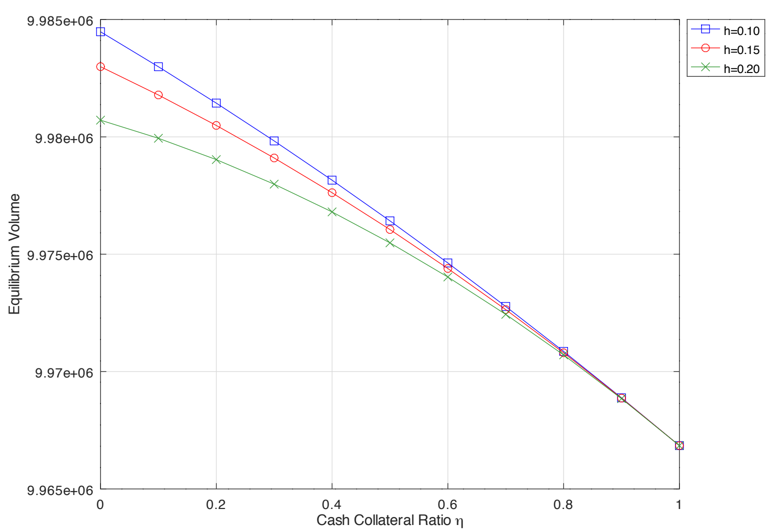

Figure 9: Cash collateral ratio η and market size of forward contract for each haircut h. Simulations are implemented for $h = 0.10, 0.15,\ 0.20$.

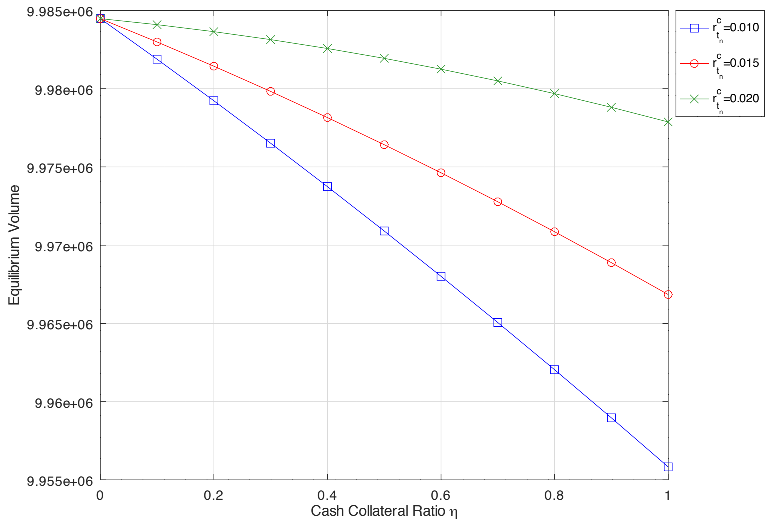

Figure 10: Cash collateral ratio η and market size of forward contract for each $r^c_{t_n}$. Simulations are implemented for $r^c_{t_n} = 0.010, 0.015,\ 0.020$.

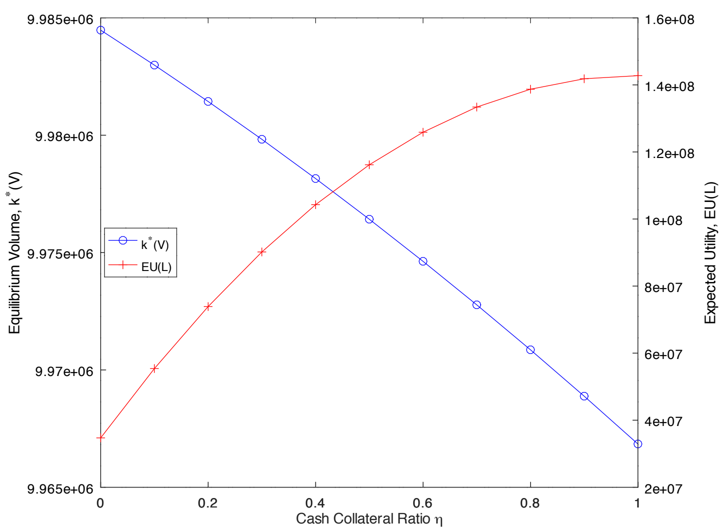

Figure 11: Cash collateral ratio η with equilibrium volume and bank's expected utility of forward contract

the option case, we investigate how the parameters, the cash collateral ratio, haircut, and collateral rate affect the forward market.

Figure 8 shows the relation between the cash collateral ratio η and equilibrium volume of the forward contract when $h = 0.50$ and $r^c_{t_n} = 0.015$. The figure also shows that the equilibrium volume is monotonically decreasing over the cash collateral ratio. The volume is largest at $\eta = 0.0$ and smallest at $\eta = 1.0$. Therefore, the market prefers non-cash collateralization. As regards the forward contract, the participant might have a positive exposure at the next MtM date even if she/he has a negative exposure at the contract date. This enables the collateral payer to reduce the loss due to own default. In this case, the collateral payer can earn more by investing the funds sourced from the participant as contract fee, because the posted collateral can be recovered. This makes the collateral payer willing to hold the cash paid as contract fee and post the collateral with the non-cash asset.

Next, we examine the effect of haircut η on the equilibrium volume. Figure 9 shows the relationship between the cash collateral ratio η and volume of the forward contract for each haircut $h = 0.50, 0.55$, and 0.60 with $r^c_{t_n} = 0.015$. We observe that the curve of $h = 0.50$ lies in the top of all curves, and the curve of $h = 0.60$ lies at the bottom, as in the option case. That is, the haircut reduces the volume of the forward contract as the option case.

Next, we observe the impact of the collateral rate on the equilibrium volume. Figure 10 shows the results of the market size for each $r^c_{t_n} = 0.010, 0.015$, and 0.020 with $h = 0.50$. The figure shows that the curve of $r^c_{t_n} = 0.020$ lies at the top of all curves, and the curve of $r^c_{t_n} = 0.010$ lies at the bottom. That is, the collateral rate boosts the liquidity of the forward contract. The effects of the haircut and collateral rate on the equilibrium volume for the forward contract are similar to those in the option case. These results are explained in the previous section.

Finally, we analyze the influence of non-cash collateralization on the bank's expected utility $E_{t_n}[U(L_{t_n+1})]$ in the forward contract case. Figure 11 shows the graphs of the equilibrium volume (marked with "$\circ$", the corresponding axis is on the left-hand side) and expected utility (marked with "+"; the corresponding axis is on the right-hand side). From the figure, the equilibrium volume is monotonically decreasing and the expected utility level monotonically increasing, depending on the increase in cash collateral ratio η. That is, the effect of the cash collateral ratio on the market volume is quite different from that on the expected utility. This also means that the maximum market volume does not achieve the maximum expected utility. Therefore, the optimal combination of the cash and non-cash collaterals to maximize the liquidity in the forward market is inefficient in the Pareto criteria, as explained in the option case.

5 Summary

In this study, we considered the effects of non-cash collateralization on the derivatives markets. We also verified how the market prefers both cash and non-cash collateralization maximize the liquidity of derivatives transactions. Liquidity in this study is measured by the (equilibrium) volume of the derivatives contract. We introduced a risk-averse participant and considered an optimization problem for her/his capital. We then solved the problem, to obtain the equilibrium volume of the derivatives contract. We theoretically and numerically carried out sensitivity analyses to demonstrate the effects of cash and non-cash collateral combination, the haircut for non-cash collateralization, the collateral rate, and the collateral payer's balance sheet on the volumes of derivatives.

We first carried out sensitivity analysis under the non-default case as a benchmark. In this case, liquidity does not depend on the haircut for non-cash collateralization. When the risk-free rate is larger than the collateral rate (this is practically true), the contract volume increases as the proportion of cash collateral decreases. That is, the market prefers the non-cash collateralization only. Furthermore, the market prefers a higher collateral rate in order to raise the liquidity.

Next, we conducted the sensitivity analyses under a defaultable environment. We considered a forward contract and an option contract as examples of derivatives contracts, with the risk-free rate larger than the collateral rate. For the option contract, posting the collateral with both cash and non-cash assets is preferred in the market to maximize liquidity, rather than only cash or non-cash collateralization. That is, both cash and non-cash collateralization are selected for the non-defaultable case. In contrast, for the forward contract, only non-cash collateralization is preferred in the market. Overall, the introduction of non-cash collateralization can boost the volumes of both derivatives contracts. This supports the recent practical action. Moreover, for both the forward and option cases, the haircut reduces but collateral rate increases the contract volumes. These results show how the market should arrange the drivers of collateral agreements to increase the liquidity of the derivatives market.

Finally, we considered the relation between non-cash collateralization and the collateral pledger's welfare. Our results show that the optimal combination of collaterals to maximize liquidity differs from that to maximize the expected utility of the collateral payer. Since the other counterparty (i.e. the collateral receiver) is risk-neutral, the optimal combination to maximize the market liquidity is not (Pareto) efficient.

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A Calculation of Statistics

Each term of (3.8) is calculated as follows.

$$\begin{aligned}E_{t_n}[g(t_{n+1};\eta)] =&E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})+g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]\\ =&E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]+E_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}],\end{aligned}$$

$$\begin{aligned}Cov_{t_n}[B_{t_{n+1}},g(T;\eta)] =&Cov_{t_n}[B_{t_{n+1}},g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]+Cov_{t_n}[B_{t_{n+1}},g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]\\ =&E_t[B_{t_{n+1}}g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]-E_{t_n}[B_{t_{n+1}}]E_t[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]\\ &+E_{t_n}[B_{t_{n+1}}g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]-E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}],\end{aligned}$$

$$\begin{aligned}Var_{t_n}[g(t_{n+1};\eta)] =&Var_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]+Var_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]\\ &+2Cov_t[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}}),g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]\\ =&E_{t_n}[(g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}}))^2]-E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]^2\\ &+E_{t_n}[(g_1(t_{n+1};\eta)1_{\tau<t_{n+1}})^2]-E_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]^2\\ &+2(E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]\\ &-E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]E_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}])\\ =&E_{t_n}[g_0(t_{n+1};\eta)^2(1-1_{\tau<t_{n+1}})]-E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]^2\\ &+E_{t_n}[g_1(t_{n+1};\eta)^2 1_{\tau<t_{n+1}}]-E_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}]^2\\ &-2E_{t_n}[g_0(t_{n+1};\eta)(1-1_{\tau<t_{n+1}})]E_{t_n}[g_1(t_{n+1};\eta)1_{\tau<t_{n+1}}],\end{aligned}$$

since $(1-1_{\tau<t_{n+1}})^2=1-1_{\tau<t_{n+1}}$, $(1_{\tau<t_{n+1}})^2=1_{\tau<t_{n+1}}$ and $(1-1_{\tau<t_{n+1}})\times 1_{\tau<t_{n+1}}=0$.

For the constant intensity process λ defined above, each term in (3.8) is calculated as follows.

$$\begin{aligned}E_{t_n}[g(t_{n+1};\eta)] =&E_{t_n}[g_0(t_{n+1};\eta)1_{\tau>t_{n+1}}]+E_{t_n}[g_1(t_{n+1};\eta)(1-1_{\tau>t_{n+1}})]\\ =&E_{t_n}[E[g_0(t_{n+1};\eta)1_{\tau>t_{n+1}}|\mathcal{F}_t^W]]+E_{t_n}[E[g_1(t_{n+1};\eta)(1-1_{\tau>t_{n+1}})|\mathcal{F}_t^W]]\\ =&E_{t_n}[g_0(t_{n+1};\eta)E[1_{\tau>t_{n+1}}|\mathcal{F}_t^W]]+E_{t_n}[g_1(t_{n+1};\eta)E[(1-1_{\tau>t_{n+1}})|\mathcal{F}_t^W]]\\ =&E_{t_n}[e^{-\lambda\Delta t}g_0(t_{n+1};\eta)]+E_{t_n}[(1-e^{-\lambda\Delta t})g_1(t_{n+1};\eta)]\\ =&e^{-\lambda\Delta t}E_{t_n}[g_0(t_{n+1};\eta)]+(1-e^{-\lambda\Delta t})E_{t_n}[g_1(t_{n+1};\eta)],\end{aligned}$$

$$\begin{aligned}Cov_{t_n}[B_{t_{n+1}},g(t_{n+1};\eta)] =&E_{t_n}[B_Tg_0(t_{n+1};\eta)1_{\tau>t_{n+1}}]-E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_0(t_{n+1};\eta)1_{\tau>t_{n+1}}]\\ &+E_{t_n}[B_{t_{n+1}}g_1(t_{n+1};\eta)(1-1_{\tau>t_{n+1}})]\\ &-E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_1(t_{n+1};\eta)(1-1_{\tau>t_{n+1}})]\\ =&E_{t_n}[e^{-\lambda\Delta t}B_{t_{n+1}}g_0(t_{n+1};\eta)]-E_{t_n}[B_{t_{n+1}}]E_{t_n}[e^{-\lambda\Delta t}g_0(t_{n+1};\eta)]\\ &+E_{t_n}[(1-e^{-\lambda\Delta t})B_{t_{n+1}}g_1(t_{n+1};\eta)]\\ &-E_{t_n}[B_{t_{n+1}}]E_{t_n}[(1-e^{-\lambda\Delta t})g_1(t_{n+1};\eta)]\\ =&e^{-\lambda\Delta t}E_{t_n}[B_{t_{n+1}}g_0(t_{n+1};\eta)]-e^{-\lambda\Delta t}E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_0(t_{n+1};\eta)]\\ &+(1-e^{-\lambda\Delta t})E_{t_n}[B_{t_{n+1}}g_1(t_{n+1};\eta)]\\ &-(1-e^{-\lambda\Delta t})E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_1(t_{n+1};\eta)]\\ =&e^{-\lambda\Delta t}\{E_{t_n}[B_{t_{n+1}}g_0(t_{n+1};\eta)]-E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_0(t_{n+1};\eta)]\}\\ &+(1-e^{-\lambda\Delta t})\{E_{t_n}[B_{t_{n+1}}g_1(t_{n+1};\eta)]-E_{t_n}[B_{t_{n+1}}]E_{t_n}[g_1(t_{n+1};\eta)]\},\end{aligned}$$

$$
\begin{aligned}
Var_{t_n}[g(t_{n+1};\eta)] =& E_{t_n}[g_0(t_{n+1};\eta)^2 1_{\tau>t_{n+1}}] - E_{t_n}[g_0(t_{n+1};\eta)1_{\tau>t_{n+1}}]^2 \\
&+ E_{t_n}[g_1(t_{n+1};\eta)^2(1-1_{\tau>t_{n+1}})] - E_{t_n}[g_1(t_{n+1};\eta)(1-1_{\tau>t_{n+1}})]^2 \\
&- 2E_{t_n}[g_0(t_{n+1};\eta)1_{\tau>t_{n+1}}]E_{t_n}[g_1(t_{n+1};\eta)(1-1_{\tau>t_{n+1}})] \\
=& e^{-\lambda\Delta t}E_{t_n}[g_0(t_{n+1};\eta)^2] - e^{-\lambda\Delta t}E_{t_n}[g_0(t_{n+1};\eta)]^2 \\
&+ (1-e^{-\lambda\Delta t})E_{t_n}[g_1(t_{n+1};\eta)^2] - (1-e^{-\lambda\Delta t})E_{t_n}[g_1(t_{n+1};\eta)]^2 \\
&- 2e^{-\lambda\Delta t}(1-e^{-\lambda\Delta t})E_{t_n}[g_0(t_{n+1};\eta)]E_{t_n}[g_1(t_{n+1};\eta)] \\
=& e^{-\lambda\Delta t}\{E_{t_n}[g_0(t_{n+1};\eta)^2] - E_{t_n}[g_0(t_{n+1};\eta)]^2\} \\
&+ (1-e^{-\lambda\Delta t})\{E_{t_n}[g_1(t_{n+1};\eta)^2] - E_{t_n}[g_1(t_{n+1};\eta)]^2\} \\
&- 2e^{-\lambda\Delta t}(1-e^{-\lambda\Delta t})E_{t_n}[g_0(t_{n+1};\eta)]E_{t_n}[g_1(t_{n+1};\eta)].
\end{aligned}
$$