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Utility Indifference Pricing and the Aumann-Serrano Performance Index

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Utility Indifference Pricing and the Aumann-Serrano Performance Index

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Abstract

A performance index based on the economic index of riskiness by Aumann and Serrano (2008) can be derived from an index based on the utility indifference price with the exponential utility function. The exponential utility function is a special utility function and relevant when the associated investor is risk averse as well as risk loving. The index based on the utility indifference price with the exponential utility function becomes an index for the random variable $\boldsymbol{g}$ of gambles with the property $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$ when the investor is risk averse and an index for the random variable $\boldsymbol{g}$ of gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ when the investor is risk loving. We provide sufficient conditions for the existence and uniqueness of the index when the investor is risk averse and risk loving.

JEL codes; G11; G32; C20; C58; C60

Keywords; Utility indifference pricing; Aumann-Serrano index; Inner rate of risk aversion; Risk loving; Risk averse

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1 Introduction

Aumann and Serrano (2008) defined an economic index of riskiness for *gambles* where gambles stand for assets, cash flows, projects, etc. with uncertain outcomes. The concept of the economic index of riskiness by Aumann and Serrano (2008) has recently received a lot of attention in the financial economics literature (cf., e.g., Foster and Hart, 2009; Hart, 2011; Homm and Pigorsch, 2012a; Kadan and Liu, 2014; Schulze, 2014). Kadan and Liu (2014) considered a performance index based on the riskiness index by Aumann and Serrano (2008) and demonstrated the Aumann and Serrano (AS) performance index can take into account high moments and disaster risk to shed new light on various issues of financial problems. In their seminal paper of the AS riskiness index, Aumann and Serrano considered the random variable $\boldsymbol{g}$ of gambles with the property $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$ and proved the existence and uniqueness of the AS riskiness index. So far, applications of the AS riskiness and performance indexes have been confined to gambles with $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$. It is fair to say that studies originating from the AS index have been quite successful.

Certainly there exist plenty of these gambles. However, there also exist other gambles which do not have the above property. We consider the random variable $\boldsymbol{g}$ of gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ in this study. For example, commercial lotteries and gambles in casinos are appropriate to be categorized as gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. Projects such as oil and natural gas mining and investments in venture capital would be other examples of such gambles where it is difficult to expect to make money but there exist chances of a big success. There are also many uncertain cash flows and projects appropriate to be described as such gambles. That is, there seem to be plenty of gambles in the real world with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. It may be appropriate to call gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ speculative or highly risky since ordinarily only risk loving investors would be interested in such gambles. The AS riskiness index excludes the important class of these gambles. The main reason why the AS riskiness index fails to take into account gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ is that they only consider risk-averse investors. However, there are risk loving investors who are willing to take risks in such highly risky gambles.

Then, it seems relevant to provide an index of such gambles. In fact, a study of gambles with negative expected value was suggested in Aumann and Serrano (2008) as a topic of future research. So far, however, to the best of our knowledge no studies have been undertaken of gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. Any proper index of gambles should treat not only the class of gambles with $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$ but also the class of gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. In this paper, we aim to fill the gap in the literature and provide an index for gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. In fact, *gambles* seem to be more often meant to be cash flows with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$, such as commercial lotteries and games in casinos in the real world. Therefore, it seems appropriate to extend gambles to include those with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ in order to make the index more practical. In this study, we intend to consider such gambles from a different point of view.

Kadan and Liu (2014) defined new performance indexes as the reciprocal of the riskiness index by Aumann and Serrano (2008) and Foster and Hart (2009). In this paper, we only consider the AS performance index, defined as the reciprocal of the riskiness index of Aumann and Serrano (2008). The AS performance index is given as the solution $P^{AS}(\boldsymbol{g})$ of the following implicit equation given by

$$E[e^{-P^{AS}(\boldsymbol{g})\boldsymbol{g}}] = 1$$

for a gamble $\boldsymbol{g}$. Kadan and Liu (2014) considered only a positive $P^{AS}(\boldsymbol{g})$ as the original AS riskiness index $1/P^{AS}(\boldsymbol{g})$ of Aumann and Serrano (2008) was defined in positive region.

Although the AS riskiness index has had a significant influence on subsequent studies in the financial economics literature, its connection has received little attention with other methods of evaluation such as certainty equivalence and utility indifference pricing in the literature of the so-called expected-utility approach. These methods are known as promising methods of evaluation of gambles in the expected utility approach (cf., e.g., Carmona, 2009). Miyahara (2014) independently proposed a performance index named as the inner rate of risk aversion (IRRA) that makes the utility indifference price with the exponential utility function $u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$ zero where α denotes the degree of risk aversion. The IRRA of a gamble $\boldsymbol{g}$ is given by the solution α_0 of the implicit equation

$$-\frac{1}{\alpha_0}\ln E[e^{-\alpha_0 \boldsymbol{g}}] = 0.$$

The utility indifference price of a gamble $\boldsymbol{g}$ is defined to be the solution ν of the equation $E[u(-\nu + \boldsymbol{g})] = u(0) = 0$ where $u(\cdot)$ denotes a utility function and E denotes expectation. When the utility function $u(\cdot)$ is increasing, if $\boldsymbol{g}_1 \geqq \boldsymbol{g}_2$, then $\nu_1 \geqq \nu_2$ where ν_i denotes the utility indifference price of $\boldsymbol{g}_i$ (i= 1,2) where $\boldsymbol{g}_i$ (i= 1,2) denotes a gamble. The property of $\nu_1 \geqq \nu_2$ if $\boldsymbol{g}_1 \geqq \boldsymbol{g}_2$ is called monotonicity, which a suitable evaluation function should satisfy. Hence, the utility indifference price is a value measure or a suitable evaluation function of gambles taking values corresponding to the value of gambles as we show below. The utility indifference price of a gamble $\boldsymbol{g}$ with the above exponential utility function is easily seen to be given by

$$-\frac{1}{\alpha}\ln E[e^{-\alpha \boldsymbol{g}}]$$

since

$$\begin{aligned} E[u(-\nu + \boldsymbol{g})] &= \frac{1}{\alpha}(1 - E[e^{-\alpha(-\nu + \boldsymbol{g})}]) \\ &= \frac{1}{\alpha}(1 - e^{\alpha\nu}E[e^{-\alpha \boldsymbol{g}}]). \end{aligned}$$

We remark the above exponential utility function is the only utility function among $\boldsymbol{C}^2$-class of utility functions under certain conditions (cf. Theorem 3.2.8 of Rolski et al., 1999 for its proof and Proposition 2 of this study given below), where $\boldsymbol{C}^2$-class is the class of functions that possess continuous second derivatives. Therefore, the exponential utility function $u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$ is a special utility function. Miyahara (2010) proved the utility indifference price, in general, satisfies a desirable criterion called a *concave monetary value measure* when the utility function is increasing and concave. And he named the above utility indifference price with the exponential utility function a *risk-sensitive value measure* (RSVM) since it is sensitive to loss of the underlying gamble. The RSVM is equivalent to certainty equivalence and an evaluation by expected utility (cf. Miyahara, 2017) when the underlying utility function is $u(x) = \frac{1}{\alpha}(1 - e^{-\alpha x})$ where certainty equivalence of $\boldsymbol{g}$ is defined by the solution $c(\boldsymbol{g})$ of the implicit equation $u(c(\boldsymbol{g})) =$

$E[u(\boldsymbol{g})]$. Therefore, the RSVM provides an evaluation conformable to expected utility, which implies the evaluation by the RSVM is desirable and proper. The method of evaluating assets, cash flows, projects, etc. with uncertain outcomes using the RSVM was introduced by Miyahara (2010) and developed in Miyahara (2014, 2017). Miyahara restricted the degree of risk aversion α to be positive when he considered the IRRA and RSVM in Miyahara (2010, 2014) since he was only concerned with risk-averse investors, where the underlying utility function is given by the above exponential utility function. However, the above exponential function continues to be a utility function when the degree of risk aversion α is negative, i.e., when an investor is risk loving. The exponential utility function is a convex utility function and hence its associated investor is risk loving when $\alpha < 0$. And it is a concave utility function and hence its associated investor is risk averse when $\alpha > 0$. The utility indifference price is to be defined for a utility function regardless of its risk preference, i.e., whether an investor is risk averse or risk loving. Thus it is quite natural to extend the Miyahara's performance index IRRA originally defined when $\alpha > 0$ to when $\alpha < 0$. The IRRA based on the utility indifference price with the same exponential utility function when the degree of risk aversion α is negative, i.e., when the associated investor is risk loving, is the one we consider in this study as the performance index of gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. The IRRA continues to be a performance index of gambles when $\alpha < 0$ as we show in this study. The IRRA was already applied in some areas (cf., Ban et al., 2016; Furukawa et al., 2018; Hodoshima, 2019) assuming the underlying investor to be risk averse. One may call the utility indifference price with the exponential utility function when the degree of risk aversion is negative a *profit-sensitive value measure* since it is sensitive to gain of the underlying gamble instead of loss of the underlying gamble.

A sufficient condition of the existence and uniqueness of the positive IRRA given by Miyahara (2014), where the underlying investor is assumed to be risk averse, is given as follows:

The moment-generating function (MGF) $E[e^{t\boldsymbol{g}}]$ *of a gamble* $\boldsymbol{g}$ *is finite for any* $t(-\infty < t < \infty)$, $E[\boldsymbol{g}] > 0$, *and* $P(\boldsymbol{g} < 0) > 0$, which is different from the sufficient condition $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$ for finite gambles (cf. Aumann and Serrano, 2008), the sufficient condition for non-finite gambles given by Homm and Pigorsch (2012b), and the necessary and sufficient condition by Schulze (2014) for no-finite gambles. We remark Miyahara did not restrict gambles to finite gambles and considered non-finite gambles which include finite gambles. The assumption of finite gambles in Aumann and Serrano (2008) is replaced by the existence of the MGF in Miyahara (2014). The condition of the existence of the MGF in Miyahara (2014) is also different from the condition of the MGF in Homm and Pigorsch (2012b) and the necessary and sufficient condition in Schulze (2014).

We provide in this paper a sufficient condition for the negative IRRA, which is obtained using the relationship between concavity and convexity, where the underlying investor is assumed to be risk loving. The sufficient condition for the negative IRRA we present in this paper is given as follows:

The MGF $E[e^{t\boldsymbol{g}}]$ *of a gamble* $\boldsymbol{g}$ *is finite for any* $t(-\infty < t < \infty)$, $E[\boldsymbol{g}] < 0$, *and* $P(\boldsymbol{g} > 0) > 0$.

We remark the IRRA depends on the gamble only, i.e., on its distribution only, not on any other parameters such as the utility function of the decision maker or his wealth. This is the same as in the AS performance index.

In the above definition of the IRRA, when the IRRA α_0 is not zero, the implicit equation of the IRRA is equivalent to the following equation:

$$E[e^{-\alpha_0 \boldsymbol{g}}] = 1,$$

which is equal to the implicit equation of the AS performance index, i.e., equation (1) of Kadan and Liu (2014). This shows the IRRA is equivalent to the AS performance index, whether α_0 is positive or negative. This is a further justification of the AS performance index. In other words, the AS performance index can be derived from the proper methods in the expected-utility approach, i.e., utility indifference pricing, certainty equivalence, and expected utility maximization, when the underlying utility function is the exponential utility function.

It is natural to extend the IRRA to the negative region since the underlying exponential utility function continues to be valid when α is negative as well as positive. As a result, we also enlarge the AS riskiness index to gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ since the AS performance index is the reciprocal of the AS riskiness index (cf. Kadan and Liu, 2014). We provide empirical examples of the negative IRRA, which indicate examples of the negative IRRA are abundant.

The rest of the paper is organized as follows. Section 2 presents definitions and properties of the RSVM, i.e., the utility indifference price with the exponential utility function. Section 3 presents a definition of the IRRA and a sufficient condition of the existence and uniqueness of the negative IRRA. Section 4 states the relationship between the IRRA and AS performance index. Section 5 presents empirical examples of the negative IRRA. Section 6 presents concluding comments.

2 Risk-Sensitive Value Measure (RSVM)

The idea of the RSVM was first introduced by Miyahara (2010), where the degree of risk aversion α was assumed to be positive ($\alpha > 0$). In this paper, we extend the idea of the RSVM to the case of $\alpha \in (-\infty, \infty)$, and the results obtained in Miyahara (2010, 2014, 2017, 2018) are extended and generalized. The proofs of the theorems in this section and next section are omitted (given in a discussion paper Hodoshima and Miyahara (2019)), because they are obtained similarly as the old proofs for the case of $\alpha > 0$ in Miyahara (2010, 2014, 2017, 2018).

2.1 Utility Indifference Price and Definition of the RSVM

We first give three definitions of utility functions.

Definition 1 (Utility Function) *A real valued function $u(x)$ defined on $(-\infty, \infty)$ is called a utility function if it satisfies the following conditions:*

1. $u(x)$ is a continuous and strictly increasing function,

2. $u(0) = 0$.

Remark 1 *The utility function is standardized to take* 0 *at* $x = 0$ *in the above definition. We adopt a weak definition of the utility function because we treat the risk-loving case as well as the risk-averse case below. We also remark that the domain of the utility function is* $(-\infty, \infty)$.

Definition 2 (Risk-Averse Utility Function) *A utility function* $u(x)$ *is called a risk-averse (or concave) utility function if it is concave.*

Utility functions of this type are associated with risk-averse investors.

Definition 3 (Risk-Loving Utility Function) *A utility function* $u(x)$ *is called a risk-loving (or convex) utility function if it is convex.*

Utility functions of this type are associated with risk-loving investors.

We next define the utility indifference price of a gamble. A gamble denotes a random variable with uncertain outcomes.

We assume that a probability space $(\Omega, \mathcal{F}, P)$ is given and that all the random variables are defined on this probability space.

Definition 4 *Let* $\boldsymbol{g}$ *be a random variable. For a utility function* $u(x)$*, the solution* v *of the following equation*

$$E[u(\boldsymbol{g} - v)] = 0 \tag{1}$$

is called the utility indifference price of $\boldsymbol{g}$.

Next, we restrict a set of random variables we treat in this paper to satisfy the following property.

Definition 5 $\mathbf{L} = \{\boldsymbol{g} : E\left[e^{t\boldsymbol{g}}\right]$ *is finite for* $-\infty < t < \infty\}$

In the above definition, $E\left[e^{t\boldsymbol{g}}\right]$ is of course the MGF of $\boldsymbol{g}$. The space $\mathbf{L}$ has the following desirable property.

Theorem 1 $\mathbf{L}$ *is a linear space.*

We restrict $\boldsymbol{g}$ to be in the space $\mathbf{L}$ in our discussions below. Our first result for the existence of the utility indifference price is given as follows.

Theorem 2 *The utility indifference price is unique if it exists.*

This result follows easily from the strictly increasing property of a utility function.

We here introduce a special utility function, the exponential function $u^{(\alpha)}(x)$. The reason why we focus our attention on the exponential function is explained in §2.2.

Definition 6 *The following function $u^{(\alpha)}(x)$ defined on $(-\infty, \infty)$,*

$$u^{(\alpha)}(x) = \begin{cases} \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right), & \alpha \neq 0, \\ x, & \alpha = 0, \end{cases} \tag{2}$$

is called the exponential utility function with the degree of risk aversion α.

Then we obtain the following theorems.

Theorem 3 *(1) The function $u^{(\alpha)}(x)$ is a continuous function of two variables (α, x).*
(2) For a fixed α, $u^{(\alpha)}(x)$ is a utility function.
(3) For $\alpha > 0$, $u^{(\alpha)}(x)$ is a strictly concave function.
(4) For $\alpha < 0$, $u^{(\alpha)}(x)$ is a strictly convex function.
(5) The coefficient of absolute risk aversion defined by Arrow (1971) and Pratt (1964) of $u^{(\alpha)}(x)$ is α.
(6) If $-\infty < \alpha_1 < \alpha_2 < \infty$, then $u^{(\alpha_1)}(x) > u^{(\alpha_2)}(x)$ for $x \neq 0$.

Theorem 4 *(1) For the exponential utility function $u^{(\alpha)}(x)$, $\alpha \in (-\infty, \infty)$, the utility indifference price of $\boldsymbol{g} \in \mathbf{L}$ exists and is unique.*
(2) The explicit form of it is

$$-\frac{1}{\alpha} \ln\left(E\left[e^{-\alpha \boldsymbol{g}}\right]\right), \quad for\ \alpha \neq 0, \tag{3}$$

and

$$E[\boldsymbol{g}], \quad for\ \alpha = 0. \tag{4}$$

Based on this theorem, we give the definition of the RSVM $U^{(\alpha)}(\boldsymbol{g})$.

Definition 7 *For $\alpha \in (-\infty, \infty)$ and $\boldsymbol{g} \in \mathbf{L}$, the following value*

$$U^{(\alpha)}(\boldsymbol{g}) = \begin{cases} -\frac{1}{\alpha} \ln \left(E\left[e^{-\alpha \boldsymbol{g}} \right] \right), & \alpha \neq 0, \\ E[\boldsymbol{g}], & \alpha = 0. \end{cases} \tag{5}$$

is called the risk-sensitive value of $\boldsymbol{g}$ with the degree of risk aversion α, and the functional $U^{(\alpha)}(\cdot)$ is called the risk-sensitive value measure (RSVM) with the degree of risk aversion α.

Remark 2 *The RSVM $U^{(\alpha)}(\boldsymbol{g})$ of $\boldsymbol{g}$ is equal to certainty equivalence $c^{(\alpha)}(\boldsymbol{g})$, where $u^{(\alpha)}(c^{(\alpha)}(\boldsymbol{g})) = E[u^{(\alpha)}(\boldsymbol{g})]$. Therefore, the evaluation by the RSVM is equivalent to the evaluation by expected utility of the exponential utility function.*

2.2 Properties of the RSVM

We first investigate the properties of the function $U^{(\alpha)}(\boldsymbol{g})$ of $\alpha \in (-\infty, \infty)$ for a fixed $\boldsymbol{g} \in \mathbf{L}$.

Theorem 5 *Let $\boldsymbol{g}$ be a random variable in $\mathbf{L}$. Then $U^{(\alpha)}(\boldsymbol{g})$ has the following properties:*
(1) *$U^{(\alpha)}(\boldsymbol{g})$ is a continuous function of $\alpha \in (-\infty, \infty)$.*
(2) *If $\boldsymbol{g}$ is not constant (i.e., $P(\boldsymbol{g} \neq E[\boldsymbol{g}]) > 0$), then $U^{(\alpha)}(\boldsymbol{g})$ is a strictly decreasing function of α, i.e.,*

$$U^{(\alpha_1)}(\boldsymbol{g}) > U^{(\alpha_2)}(\boldsymbol{g}), \quad for\ \alpha_1 < \alpha_2. \tag{6}$$

Before we explain the properties of the functional $U^{(\alpha)}(\boldsymbol{g})$ of $\boldsymbol{g} \in \mathbf{L}$ for a fixed $\alpha \in (-\infty, \infty)$, we need to prepare some concepts.

Definition 8 (concave monetary value measure) *A function $v(\cdot)$ defined on a linear space $\mathbf{L}$ of random variables is called* a concave monetary value measure (or concave monetary utility function) *on $\mathbf{L}$ if it satisfies the following conditions:*
(i) (Normalization) : $v(\mathbf{0}) = 0$,
(ii) (Monetary property) : $v(\boldsymbol{g} + m) = v(\boldsymbol{g}) + m$, where m is non-random,
(iii) (Monotonicity) : (a) If $\boldsymbol{g}_1 \geqq \boldsymbol{g}_2$, i.e., $P(\boldsymbol{g}_1 \geqq \boldsymbol{g}_2) = 1$, then $v(\boldsymbol{g}_1) \geqq v(\boldsymbol{g}_2)$,
(b) If $\boldsymbol{g}_1 \geqq \boldsymbol{g}_2$ and $P(\boldsymbol{g}_1 > \boldsymbol{g}_2) > 0$, then $v(\boldsymbol{g}_1) > v(\boldsymbol{g}_2)$,
(iv) (Concavity) : $v(\lambda \boldsymbol{g}_1 + (1-\lambda)\boldsymbol{g}_2) \geqq \lambda v(\boldsymbol{g}_1) + (1-\lambda) v(\boldsymbol{g}_2) \quad$ for $0 \leqq \lambda \leqq 1$,
(v) (Law invariance) : $v(\boldsymbol{g}_1) = v(\boldsymbol{g}_2)$ whenever $\mathrm{law}(\boldsymbol{g}_1) = \mathrm{law}(\boldsymbol{g}_2)$,

The concept of a concave monetary value measure (or concave monetary utility function) was introduced in Cheridito et al. (2006).

Remark 3 $v(m) = m$ *follows from* (i) and (ii).

Remark 4 *Set* $\lambda = 1/2$ *and put* $\boldsymbol{g}_1 = \boldsymbol{g}$ *and* $\boldsymbol{g}_2 = -\boldsymbol{g}_1 = -\boldsymbol{g}$ *in the concavity condition. Then we have*

$$v(\frac{1}{2}\boldsymbol{g} + (1 - \frac{1}{2})(-\boldsymbol{g})) \geqq \frac{1}{2}v(\boldsymbol{g}) + (1 - \frac{1}{2})v(-\boldsymbol{g}).$$

The left hand side of the above inequality is equal to 0*. Hence, we have*

$$v(\boldsymbol{g}) \leqq -v(-\boldsymbol{g}).$$

When $v(\boldsymbol{g}) > 0$*, the above inequality implies* $v(-\boldsymbol{g}) < 0$ *and* $|v(-\boldsymbol{g})| \geqq v(\boldsymbol{g})$*. This implies the investor who obeys the concave monetary value measure is more sensitive to the loss of* $\boldsymbol{g}$ *being negative than the gain of* $\boldsymbol{g}$ *being positive.*

Definition 9 (convex monetary value measure) *A function* $v(\cdot)$ *defined on a linear space* $\mathbf{L}$ *of random variables is called* a convex monetary value measure (or convex monetary utility function) *on* $\mathbf{L}$ *if it satisfies the following conditions:*
(i) (Normalization) : $v(\mathbf{0}) = 0$,
(ii) (Monetary property) : $v(\boldsymbol{g} + m) = v(\boldsymbol{g}) + m$, where m is non-random,
(iii) (Monotonicity) : (a) If $\boldsymbol{g}_1 \geqq \boldsymbol{g}_2$, i.e., $P(\boldsymbol{g}_1 \geqq \boldsymbol{g}_2) = 1$, then $v(\boldsymbol{g}_1) \geqq v(\boldsymbol{g}_2)$,
(b) If $\boldsymbol{g}_1 \geqq \boldsymbol{g}_2$ and $P(\boldsymbol{g}_1 > \boldsymbol{g}_2) > 0$, then $v(\boldsymbol{g}_1) > v(\boldsymbol{g}_2)$,
(iv) (Convexity) : $v(\lambda\boldsymbol{g}_1 + (1 - \lambda)\boldsymbol{g}_2) \leqq \lambda v(\boldsymbol{g}_1) + (1 - \lambda)v(\boldsymbol{g}_2)$ for $0 \leqq \lambda \leqq 1$,
(v) (Law invariance) : $v(\boldsymbol{g}_1) = v(\boldsymbol{g}_2)$ whenever $\mathrm{law}(\boldsymbol{g}_1) = \mathrm{law}(\boldsymbol{g}_2)$,

Remark 5 *Similarly as Remark 4, it is shown that the investor who obeys the convex monetary value measure is more sensitive to the gain of* $\boldsymbol{g}$ *being positive than the loss of* $\boldsymbol{g}$ *being negative.*

Now we can state essential properties of the RSVM. The first theorem is given as follows.

Theorem 6 *For $\alpha > 0$, the RSVM $U^{(\alpha)}(\cdot)$ is a concave monetary value measure.*

Remark 6 *This result was proved in a more general form in Miyahara(2010) as follows.*

Proposition 1 (Proposition 2 of Miyahara(2010)) *For a risk-averse utility function $u(x)$, the utility indifference value measure $v(\cdot)$ of $u(x)$ is a concave monetary value measure.*

The second theorem is given as follows.

Theorem 7 *For $\alpha < 0$, the RSVM $U^{(\alpha)}(\cdot)$ is a convex monetary value measure.*

We here introduce a concept relating to the value measure.

Definition 10 (Independence-Additivity) *If a value measure $v(\cdot)$ satisfies the following condition*

(independence-additivity): $v(\boldsymbol{g}_1 + \boldsymbol{g}_2) = v(\boldsymbol{g}_1) + v(\boldsymbol{g}_2)$ *if $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are independent, then $v(\cdot)$ is said to have the independence-additivity property.*

This property is desirable for the project evaluation functional, especially when we want to evaluate portfolios of projects.

The following theorem is easily proved.

Theorem 8 *The RSVM $U^{(\alpha)}(\boldsymbol{g})$ has the independence-additivity property.*

The converse of this theorem was given by Rolski et al. (1999). We present it as follows.

Proposition 2 *Let $v(\boldsymbol{g})$ be a utility indifference price of $\boldsymbol{g}$ determined by a utility function $u(x)$ which is of $C^{(2)}$-class, increasing, concave (convex), and normalized as $u(0) = 0$, $u'(0) = 1$, and $u''(0) = -\alpha$. Then, if $v(\boldsymbol{g})$ has the independence-additivity property, $u(x)$ is of the following form*

$$u(x) = u^{(\alpha)}(x) = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right). \tag{7}$$

The above proposition implies the exponential utility function and the RSVM are respectively the only utility function and the only utility indifference value measure among $C^{(2)}$-class of utility functions that have the independence-additivity property.

2.3 Characterization of the RSVM

The properties of RSVM $U^{(\alpha)}(\boldsymbol{g})$ for positive α have been investigated in Miyahara(2010, 2014, 2017, 2018). We summarize the properties of RSVM obtained there:

(1) The RSVM is a concave monetary value measure.

(2) The RSVM is the utility indifference price of the exponential utility function, and it has the risk aversion parameter α.

(3) By the use of the RSVM, the scale risk and optimal scale of a project can be discussed.

(4) The RSVM has the independence-additivity property, and the RSVM is almost the only one with this property in the class of all utility indifference prices.

(5) The dynamic RSVM has the time-consistency property, and the RSVM is almost the only one with this property in the class of all utility indifference prices.

These properties are all desirable properties that a reasonable value measure should satisfy. And we would like to emphasize that the RSVM is the only value measure with all the properties stated above, in the class of utility indifference value measures. Hence, the RSVM is an excellent and exceptional value measure.

Remark 7 *The above properties of the RSVM are easily extended to the case of* $-\infty < \alpha < \infty$ *in a natural way.*

3 Inner Rate of Risk Aversion (IRRA)

The idea of the IRRA was introduced first in Miyahara (2014), and the case of $\alpha \geqq 0$ was investigated in Miyahara (2018) a little more precisely.

3.1 Definition and Existence of the IRRA

We first give the definition of the IRRA.

Definition 11 *Let* $\boldsymbol{g}$ *be a random variable in* $\mathbf{L}$. *Then the solution* α_0 *of the following equation*

$$U^{(\alpha)}(\boldsymbol{g}) = 0 \tag{8}$$

is called the IRRA of $\boldsymbol{g}$ *and denoted by* $\alpha_0(\boldsymbol{g})$.

Let α be the degree of risk aversion of an investor, and suppose that $\boldsymbol{g}$ is a non-deterministic asset. Then, from Theorem 5 (2), $U^{(\alpha)}(\boldsymbol{g})$ is a strictly decreasing function of α. Therefore, if $\alpha < \alpha_0(\boldsymbol{g})$, then $U^{(\alpha)}(\boldsymbol{g}) > U^{(\alpha_0(\boldsymbol{g}))}(\boldsymbol{g}) = 0$ and the investor with the degree of risk aversion α is less risk averse than the investor with $\alpha_0(\boldsymbol{g})$ and accepts $\boldsymbol{g}$. On the other hand, if $\alpha > \alpha_0(\boldsymbol{g})$, then $U^{(\alpha)}(\boldsymbol{g}) < U^{(\alpha_0(\boldsymbol{g}))}(\boldsymbol{g}) = 0$ and the investor with the degree of risk aversion α is more risk averse than the investor with $\alpha_0(\boldsymbol{g})$ and rejects $\boldsymbol{g}$. From the above consideration, we can say that if $\alpha_0(\boldsymbol{g}_1) < \alpha_0(\boldsymbol{g}_2)$, then $\alpha_0(\boldsymbol{g}_2)$ is more valuable than $\alpha_0(\boldsymbol{g}_1)$ in the sense of the RSVM.

As a corollary of Theorem 5, we obtain the following theorem.

Theorem 9 *For a not constant $\boldsymbol{g}$ in $\mathbf{L}$ (i.e., $P(\boldsymbol{g} \neq E[\boldsymbol{g}]) > 0$), the IRRA of $\boldsymbol{g}$ is uniquely defined if it exists.*

Remark 8 *Since $U^{(\alpha)}(\mathbf{0}) = 0, \quad -\infty < \alpha < \infty$, $\alpha_0(\mathbf{0})$ is not defined uniquely. But it is convenient to define as*

$$\alpha_0(\mathbf{0}) = 0. \tag{9}$$

For the existence of the IRRA, we have the following result.

Theorem 10 *Assume that a random variable $\boldsymbol{g} \in \mathbf{L}$ satisfies the following conditions*

$$E[\boldsymbol{g}] > 0 \quad \text{and} \quad P(\boldsymbol{g} < 0) > 0. \tag{10}$$

Then the IRRA $\alpha_0(\boldsymbol{g})$ of $\boldsymbol{g}$ exists uniquely and is positive.

Theorem 11 *Assume that a random variable $\boldsymbol{g} \in \mathbf{L}$ satisfies the following conditions*

$$E[\boldsymbol{g}] < 0 \quad \text{and} \quad P(\boldsymbol{g} > 0) > 0. \tag{11}$$

Then the IRRA $\alpha_0(\boldsymbol{g})$ of $\boldsymbol{g}$ exists uniquely and is negative.

3.2 Properties of the IRRA

As a value functional on the space $\mathbf{L}$, the IRRA $\alpha_0(\cdot)$ has the following natural property.

Theorem 12 *Let $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ be random variables in $\mathbf{L}$ and assume that $\alpha_0(\boldsymbol{g}_1)$ and $\alpha_0(\boldsymbol{g}_2)$ exist. Then, if $\boldsymbol{g}_1 \leqq \boldsymbol{g}_2$ and $P(\boldsymbol{g}_1 < \boldsymbol{g}_2) > 0$, it holds that $\alpha_0(\boldsymbol{g}_1) < \alpha_0(\boldsymbol{g}_2)$.*

From Theorem 12 it follows that the IRRA has the monotonicity property.

Remark 9 *See Miyahara (2018) for the details of the properties of IRRA in the case of* $\alpha \geqq 0$.

4 Relationship between the IRRA and AS performance index

We have established the existence and uniqueness of the negative IRRA in the previous section when the degree of risk aversion is negative, i.e., for gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. The negative IRRA of a gamble $\boldsymbol{g}$ is the solution $\alpha_0(< 0)$ for the implicit equation

$$U^{(\alpha_0)}(\boldsymbol{g}) \equiv -\frac{1}{\alpha_0} \ln E[e^{-\alpha_0 \boldsymbol{g}}] = 0.$$

When the negative α_0 exists in the above equation, it is also the solution of the following implicit equation

$$E[e^{-\alpha_0 \boldsymbol{g}}] = 1.$$

The solution α_0 of the above implicit equation is the AS performance index (cf., Aumann and Serrano, 2008 and Kadan and Liu, 2014). Therefore, the existence and uniqueness of the negative IRRA are equivalent to those of the negative AS performance index. Hence, our extension of the IRRA when the degree of risk aversion is negative results in extension of the AS performance index when the degree of risk aversion is negative. The same applies to the equivalence of the positive IRRA and positive AS performance index when the degree of risk aversion is positive, the case the original IRRA and AS performance indexes treated. Hence, the IRRA is equivalent to the AS performance index when the degree of risk aversion is negative as well as positive.

The original AS riskiness index and hence the original AS performance index restrict the underlying investor to be risk averse. However, letting the AS riskiness and performance indexes negative makes the underlying investor risk loving. We have the following equation by rewriting the implicit equation for the AS performance index:

$$\frac{1}{\alpha_0}(1 - E[e^{-\alpha_0 (w_0 + \boldsymbol{g})}]) = \frac{1}{\alpha_0}(1 - e^{-\alpha_0 w_0}),$$

for some initial wealth w_0. The two implicit equations for the AS performance index are equivalent regardless of w_0. Therefore, α_0 is the level of absolute risk aversion that makes an investor with the exponential utility function indifferent between taking $\boldsymbol{g}$ and the status quo, regardless of the initial wealth w_0 (cf. Kadan and Liu, 2014), when α_0 is positive as well as negative. Then, the level of α_0 continues to be an index of performance when α_0 is both positive and negative. Therefore, the AS index continues to be an index of riskiness or performance when it is both positive and negative.

The AS riskiness index is derived from the axioms of *duality* and *positive homogeneity* (cf. Theorem A of Aumann and Serrano, 2008). Aumann and Serrano showed that the AS index is related to *constant absolute risk aversion* with an exponential utility function $u(x) = -e^{-\alpha x}$, where the coefficient of absolute risk aversion does not depend on the investor's wealth (cf. Theorem B of Aumann and Serrano, 2008). On the other hand, the RSVM and IRRA begin with the utility indifference price of the exponential utility function $u(x) = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right)$. The idea of the IRRA is straightforward and the IRRA is naturally defined when the investor is risk averse and risk loving.

5 Empirical Examples of the negative IRRA (AS performance index)

In this section, we present three empirical examples of the negative IRRA or negative AS performance index. We derive the IRRA or AS performance index using the generalized method of moments estimation (GMME) as in Kadan and Liu (2014), assuming each data to be realizations of a random sample. We obtain the GMME by grid search. We pick up assets with negative means as candidates of the negative IRRA or AS performance index. We have easily succeeded in obtaining the negative IRRA or negative AS performance index by finding assets with negative means.

We first employ daily return data of the crude oil WTI future price in percentages from January 2, 2008 to December 29, 2017, the period of covering the financial crisis of 2008-2009 up to the end of 2017. Summary statistics of the return data are given as follows: mean -0.018, standard deviation 2.475, skewness 0.159, and kurtosis 7.648. Mean, an estimate of expectation of the underlying distribution, being negative indicates part of the sufficient condition of the existence and uniqueness of the negative IRRA is satisfied. In this case, the IRRA is -0.006. The AS performance index is the same.

Next, we employ daily return data of the gold future closing price in percentages from January 2, 2013 to December 31, 2015. Summary statistics of its return data are given as follows: mean -0.061, standard deviation 0.040, skewness -1.046, and kurtosis 13.083. In this case, the IRRA or AS performance index is -0.102.

The third data is daily return data of the IBM daily closing price in percentages from October 2, 2012 to April 28, 2017. Summary statistics of its return data are given as follows: mean -0.024, standard deviation 0.035, skewness -1.096, and kurtosis 9.851. Again mean is negative. In this case, the IRRA or AS performance index is -0.034.

These three empirical examples indicate gambles with the negative IRRA or negative AS performance index are abundant in the real world. Hence, use of the IRRA or AS performance index would be quite useful as an index of evaluating gambles such as assets, cash flows, projects, etc. with uncertain outcomes when gambles $\boldsymbol{g}$ have not only the property with $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$ but also the property with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$.

6 Concluding comments

Aumann and Serrano (2008) proposed the economic index of riskiness for gambles. The riskiness index of Aumann and Serrano has recently received a lot of attention in the financial economics literature. However, the index is defined for the gamble $\boldsymbol{g}$ with the property $E[\boldsymbol{g}] > 0$ and $P(\boldsymbol{g} < 0) > 0$. Although this class of gambles is abundant in the world, there are also plenty of other gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. It is desirable to provide an index for this neglected class of gambles in the literature with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$.

We have provided an index for this neglected class of gambles by extending the work of Miyahara (2014). Miyahara (2014) proposed the concept of the IRRA based on the utility indifference price with the exponential utility function as an index of showing the desirability of assets, uncertain projects, future cash flows, etc. when the degree of risk aversion is positive. Since the underlying exponential utility function continues to be valid when the degree of risk aversion is negative, i.e., when the associated investor is risk loving, the IRRA is valid as the index derived from the utility indifference price when the degree of risk aversion is both positive and negative. It is natural to extend the IRRA when the utility function is risk loving since the utility indifference price is to be defined regardless of risk preference. Any appropriate index should be in principle able to deal with both types of gambles since the neglected class of gambles is also abundant in the real world. We have proved the existence and uniqueness of the IRRA under our sufficient condition when the degree of risk aversion is negative, using the relationship between concavity and convexity. Since the IRRA can be shown to be equivalent to the AS performance index, we have consequently extended the AS performance index when the degree of risk aversion is negative, i.e., for gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$. This is a further justification of the AS performance index since it can be derived from the proper methods in the expected-utility approach, i.e., utility indifference pricing, certainty equivalence, and expected utility. As a result, we have also extended the AS riskiness index when the degree of risk aversion is negative since the riskiness index is just the reciprocal of the performance index for both the IRRA and AS performance index.

We have also shown empirical examples of the negative IRRA or negative AS performance index. These examples indicate the class of gambles with the property $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ is prevalent. Therefore, extending the IRRA or AS index to gambles with $E[\boldsymbol{g}] < 0$ and $P(\boldsymbol{g} > 0) > 0$ makes it more practical to be applicable to many gambles in reality.

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Appendix: Proofs

The proofs we provide in this paper are similar to the proofs in Miyahara (2010, 2014, 2017, 2018). Only the case $\alpha \geqq 0$ was treated in these previous papers. In this paper, we extend theorems and proofs in these previous studies to the case $\alpha \in (-\infty, \infty)$ and simplify some of the proofs in the previous studies.

A Risk-Sensitive Value Measure (RSVM)

A.1 Utility Indifference Price and the Definition of the RSVM

(Proof of Theorem 1) Let $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathbf{L}$ and $a, b \in (-\infty, \infty)$. Then, using the Schwarz's inequality, we obtain

$$E[e^{t(a\boldsymbol{g}_1+b\boldsymbol{g}_2)}] = E[e^{t(a\boldsymbol{g}_1)}e^{t(b\boldsymbol{g}_2)}] \leqq \sqrt{E[e^{2ta\boldsymbol{g}_1}]}\sqrt{E[e^{2tb\boldsymbol{g}_2}]} < \infty. \tag{12}$$

This implies Theorem 1.

(Q.E.D.)

(Proof of Theorem 2)

This result follows easily from the strictly increasing property of a utility function.

(Q.E.D.)

(Proof of Theorem 3)

(1) The continuity of $u^{(\alpha)}(x)$ at a point $(\alpha, x), \alpha \neq 0$ is trivial. The continuity of $u^{(\alpha)}(x)$ at a point $(0, x)$ follows from the fact that

$$\lim_{\alpha \to 0} u^{(\alpha)}(x) = \lim_{\alpha \to 0} \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right) = \frac{\partial}{\partial \alpha}\left(-e^{-\alpha x}\right)|_{\alpha=0} = x. \tag{13}$$

(2) Trivial.

(3) Trivial.

(4) Trivial.

(5) It is well-known.

(6) Trivial.

(Q.E.D.)

(Proof of Theorem 4)

1. The case of $\alpha \neq 0$.

The utility indifference price v of $\boldsymbol{g}$ is the solution of the following equation,

$$E\left[u^{(\alpha)}(\boldsymbol{g}-v)\right] = \frac{1}{\alpha}E\left[1-e^{-\alpha(\boldsymbol{g}-v)}\right] = \frac{1}{\alpha}\left(1-E\left[e^{-\alpha\boldsymbol{g}}\right]e^{\alpha v}\right) = 0. \tag{14}$$

From this we obtain

$$e^{\alpha v} = \left(E\left[e^{-\alpha\boldsymbol{g}}\right]\right)^{-1}, \tag{15}$$

and hence

$$v = -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha\boldsymbol{g}}\right]\right). \tag{16}$$

2. The case of $\alpha = 0$.

The defining equation of the utility indifference price of $\boldsymbol{g}$ is

$$E\left[u^{(\alpha)}(\boldsymbol{g}-v)\right] = E\left[(\boldsymbol{g}-v)\right] = E\left[\boldsymbol{g}\right] - v = 0. \tag{17}$$

So, we obtain the result $v = E\left[\boldsymbol{g}\right]$.

(Q.E.D.)

A.2 Properties of the RSVM

(Proof of Theorem 5)

(1) From the assumption of the integrability, $E\left[e^{t\boldsymbol{g}}\right] < \infty, -\infty < t < \infty$, the continuity of $U^{(\alpha)}(\boldsymbol{g})$ follows.

(2) Put $v_1 = U^{(\alpha_1)}(\boldsymbol{g})$ and $v_2 = U^{(\alpha_2)}(\boldsymbol{g})$. Since $u^{(\alpha_1)}(x) > u^{(\alpha_2)}(x)$ for $x \neq 0$, we obtain

$$0 = E\left[u^{(\alpha_1)}(\boldsymbol{g}-v_1)\right] > E\left[u^{(\alpha_2)}(\boldsymbol{g}-v_1)\right], \tag{18}$$

where we use the assumption that $\boldsymbol{g}$ is not constant. Since $u^{(\alpha_2)}(x)$ is a strictly increasing function of x, it follows from this inequality that $v_2 < v_1$.

(Q.E.D.)

(Proof of Theorem 6)

(i) trivial.

(ii) follows from the following equation

$$-\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha(\boldsymbol{g}+m)}\right]\right) = -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha\boldsymbol{g}}\right]e^{-\alpha m}\right) = -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha\boldsymbol{g}}\right]\right) + m. \qquad (19)$$

(iii) Put $v_1 = U^{(\alpha)}(\boldsymbol{g}_1)$ and $v_2 = U^{(\alpha)}(\boldsymbol{g}_2)$.

(a) The utility function $u^{(\alpha)}(x)$ is a strictly increasing function of x. So we obtain

$$0 = E\left[u^{(\alpha)}(\boldsymbol{g}_1 - v_1)\right] \geqq E\left[u^{(\alpha)}(\boldsymbol{g}_2 - v_1)\right]. \qquad (20)$$

From this inequality it follows that $v_1 \geqq v_2$.

(b) In this case, it holds that

$$0 = E\left[u^{(\alpha)}(\boldsymbol{g}_1 - v_1)\right] > E\left[u^{(\alpha)}(\boldsymbol{g}_2 - v_1)\right]. \qquad (21)$$

From this it follows that $v_1 > v_2$.

(iv) Put $v_1 = U^{(\alpha)}(\boldsymbol{g}_1)$ and $v_2 = U^{(\alpha)}(\boldsymbol{g}_2)$.

Since $u^{(\alpha)}(x)$ is a concave function of x, we obtain the following relationship

$$\begin{aligned}
& E\left[u^{(\alpha)}(\lambda\boldsymbol{g}_1 + (1-\lambda)\boldsymbol{g}_2) - (\lambda v_1 + (1-\lambda)v_2)\right] \\
= \; & E\left[u^{(\alpha)}(\lambda(\boldsymbol{g}_1 - v_1) + (1-\lambda)(\boldsymbol{g}_2 - v_2))\right] \\
\geqq \; & E\left[\lambda u^{(\alpha)}(\boldsymbol{g}_1 - v_1) + (1-\lambda)u^{(\alpha)}(\boldsymbol{g}_2 - v_2)\right] \\
= \; & \lambda E\left[u^{(\alpha)}(\boldsymbol{g}_1 - v_1)\right] + (1-\lambda)E\left[u^{(\alpha)}(\boldsymbol{g}_2 - v_2)\right] \\
= \; & 0. \qquad (22)
\end{aligned}$$

From this inequality it follows that $\lambda v_1 + (1-\lambda)v_2 \leqq v_{\{\lambda\boldsymbol{g}_1+(1-\lambda)\boldsymbol{g}_2\}}$

where $v_{\{\lambda\boldsymbol{g}_1+(1-\lambda)\boldsymbol{g}_2\}} \equiv U^{(\alpha)}(\lambda\boldsymbol{g}_1 + (1-\lambda)\boldsymbol{g}_2)$.

(Remark: If $P(\boldsymbol{g}_1 \neq \boldsymbol{g}_2) > 0$, then $\lambda v_1 + (1-\lambda)v_2 < v_{\{\lambda\boldsymbol{g}_1+(1-\lambda)\boldsymbol{g}_2\}}$ for $\lambda \neq 0, 1$.)

(v) Trivial.

(Q.E.D.)

(Proof of Proposition 1) Similarly as in the proof of Theorem 6, Proposition 1 can be proved using the assumption of the concave utility function.

(Proof of Theorem 7)

The results (i), (ii),(iii), and (v) are already proved in Theorem 6. The result (iv) is similarly proved as in the proof of (iv) of Theorem 6 by the use of convexity of $u^{(\alpha)}(x)$ instead of concavity.

(Q.E.D.)

(Proof of Theorem 8)

Assume that $\boldsymbol{g}_1, \boldsymbol{g}_2 \in \mathbf{L}$ are independent. Then

$$\begin{aligned} U^{(\alpha)}(\boldsymbol{g}_1+\boldsymbol{g}_2) &= -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha(\boldsymbol{g}_1+\boldsymbol{g}_2)}\right]\right) \\ &= -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha\boldsymbol{g}_1}\right]E\left[e^{-\alpha\boldsymbol{g}_2}\right]\right) \\ . \quad &= -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha\boldsymbol{g}_1}\right]\right) - \frac{1}{\alpha}\ln\left(E\left[e^{-\alpha\boldsymbol{g}_2}\right]\right) \\ &= U^{(\alpha)}(\boldsymbol{g}_1) + U^{(\alpha)}(\boldsymbol{g}_2). \end{aligned} \tag{23}$$

(Q.E.D.)

(Proof of Proposition 2)

See Theorem 3.2.8 of Rolski et al. (1999).

B Inner Rate of Risk Aversion (IRRA)

B.1 Definition and Existence of the IRRA

(Proof of Theorem 9)

By Theorem 5, $U^{(\alpha)}(\boldsymbol{g})$ is a strictly decreasing continuous function of α. Therefore $\alpha_0(\boldsymbol{g})$ is unique if it exists.

(Q.E.D.)

(Proof of Theorem 10)

(The idea of the proof is the same as that of the proof of Theorem 2 of Miyahara (2018).)

$U^{(0)}(\boldsymbol{g}) = E[\boldsymbol{g}] > 0$ and $U^{(\alpha)}(\boldsymbol{g})$ is a strictly decreasing continuous function of α. So what we have to prove is that

$$\lim_{\alpha\to\infty} U^{(\alpha)}(\boldsymbol{g}) < 0. \tag{24}$$

From the assumption that $P(\boldsymbol{g} < 0) > 0$, we can choose two constants $a > 0$ and $\delta > 0$ such that

$$P(\boldsymbol{g} < -a) > \delta > 0. \tag{25}$$

Then we obtain for any $\alpha > 0$

$$\begin{aligned} E[e^{-\alpha \boldsymbol{g}}] &= E[e^{-\alpha \boldsymbol{g}} 1_{(-\infty,-a)}(\boldsymbol{g})] + E[e^{-\alpha \boldsymbol{g}} 1_{[-a,\infty)}(\boldsymbol{g})] \\ &> E[e^{-\alpha \boldsymbol{g}} 1_{(-\infty,-a)}(\boldsymbol{g})] \geqq e^{\alpha a} P(\boldsymbol{g} < -a) \\ &> e^{\alpha a}\delta \end{aligned} \tag{26}$$

where $1_{(-\infty,-a)}(\boldsymbol{g}) = 1$ if $\boldsymbol{g} \in (-\infty, -a)$ and 0 otherwise and $1_{[-a,\infty)}(\boldsymbol{g}) = 1$ if $\boldsymbol{g} \in [-a, \infty)$ and 0 otherwise. Hence we have

$$U^{(\alpha)}(\boldsymbol{g}) = -\frac{1}{\alpha}\ln\left(E\left[e^{-\alpha \boldsymbol{g}}\right]\right) < -\frac{1}{\alpha}\ln\left(e^{\alpha a}\delta\right) = -a - \frac{\ln\delta}{\alpha}. \tag{27}$$

The right hand side of the above inequality tends to $-a$ when $\alpha \to \infty$, which implies $\lim_{\alpha\to\infty} U^{(\alpha)}(\boldsymbol{g}) < 0$.
(Q.E.D.)

(Proof of Theorem 11)
Put $\boldsymbol{g}^{minus} = -\boldsymbol{g}$. Then $\boldsymbol{g}^{minus}$ satisfies all the assumptions of $\boldsymbol{g}$ in Theorem 10. Therefore, $\alpha_0(\boldsymbol{g}^{minus})$ exists and it is positive. By the definition of $\alpha_0(\boldsymbol{g}^{minus})$, $U^{(\alpha_0(\boldsymbol{g}^{minus}))}(\boldsymbol{g}^{minus}) = 0$, so we have

$$-\frac{1}{\alpha_0(\boldsymbol{g}^{minus})}\ln\left(E\left[e^{-\alpha_0(\boldsymbol{g}^{minus})\boldsymbol{g}^{minus}}\right]\right) = 0 \quad \text{and} \quad \mathrm{E}\left[\mathrm{e}^{-\alpha_0(\boldsymbol{g}^{\mathrm{minus}})\boldsymbol{g}^{\mathrm{minus}}}\right] = 1. \tag{28}$$

From this we obtain

$$1 = E\left[e^{-\alpha_0(\boldsymbol{g}^{minus})\boldsymbol{g}^{minus}}\right] = E\left[e^{-\alpha_0(-\boldsymbol{g})(-\boldsymbol{g})}\right] = E\left[e^{\alpha_0(-\boldsymbol{g})\boldsymbol{g}}\right], \tag{29}$$

which implies

$$\begin{aligned} U^{(-\alpha_0(-\boldsymbol{g}))}(\boldsymbol{g}) &= -\frac{1}{-\alpha_0(-\boldsymbol{g})}\ln\left(E\left[e^{-(-\alpha_0(-\boldsymbol{g}))\boldsymbol{g}}\right]\right) \\ &= -\frac{1}{-\alpha_0(-\boldsymbol{g})}\ln\left(E\left[e^{\alpha_0(-\boldsymbol{g})\boldsymbol{g}}\right]\right) \\ &= 0. \end{aligned} \tag{30}$$

The above equality proves that $\alpha_0(\boldsymbol{g})$ exists and it is equal to $-\alpha_0(-\boldsymbol{g})$. Since $\alpha_0(-\boldsymbol{g}) = \alpha_0(\boldsymbol{g}^{minus}) > 0$, $\alpha_0(\boldsymbol{g}) = -\alpha_0(-\boldsymbol{g}) < 0$ is proved.
(Q.E.D.)

B.2 Properties of the IRRA

(Proof of Theorem 12)

By the property (iii) (Monotonicity) (b) of Theorem 6, we know

$$U^{(\alpha)}(\boldsymbol{g}_1) < U^{(\alpha)}(\boldsymbol{g}_2), \quad -\infty < \alpha < \infty. \tag{31}$$

Therefore

$$0 = U^{(\alpha_0(\boldsymbol{g}_1))}(\boldsymbol{g}_1) < U^{(\alpha_0(\boldsymbol{g}_1))}(\boldsymbol{g}_2). \tag{32}$$

Since $U^{(\alpha)}(\boldsymbol{g}_2)$ is a strictly decreasing function of α, we obtain the inequality $\alpha_0(\boldsymbol{g}_1) < \alpha_0(\boldsymbol{g}_2)$.
(Q.E.D.)