# The Existence of The Environment-Dependent Optimal (s, S) Policy

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## Abstract

Inventory systems are often subject to randomly changing exogenous environment-conditions that affect the demand for the product, the supply, and the cost structure.

This paper analyzes a single product, periodic review inventory model with uncertain demand in a random environment. The demand distribution and cost parameters depend on environmental fluctuations which is assumed to follows a discrete-time Markov chain. To minimize the total discounted expected cost (fixed ordering, ordering, holding, and shortage costs), we formulate the model as a dynamic programming problem. For the finite-horizon model, we prove that the objective function is convex and that the structure of the optimal policy is characterized by two environmental-dependent critical numbers for the initial inventory level at each period. Expressions for solving the critical numbers and the optimal planned ordering are obtained. We further show that the solution for the finite-horizon model converges to that of the infinite-horizon model.

*Keywords*: Inventory; Dynamic programming; Periodic review models

## 1. Introduction

The inventory control has long focused on managing certain specific types of probability distribution in the demand for the products. Since many models include a purely random component in the demand process, it will be difficult to describe the inventory control of following products.

**Products sensitive to economic conditions:** The demand for many products responds in part to changes in certain the basic economic variables, such as GNP or interest rates. Consequently, many demand forecasting models include submodels describing these variables; information about such variables, whether formally modeled or not, is routinely used to project future demands. However, standard inventory-control models based on purely random demand processes have no way to use such information. Such models can include time-dependent demand, but not situations where our information about future demand changes as the time elaspes.

**Products subject to obsolescence:** Many products are subject to obsolescence; that is, demand is now healthy, but there is sizeable chance that the demand will drop precipitously in the future. Obvious examples are products in industries with high rates of technical innovation, such as computers and phar-

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maceuticals. Also, products in markets with frequent shifts in consumer tastes fit this pattern; including books, CD, perfumes, and some food items. Both the timing and degree of obsolescence are typically uncertain. The standard models with uncertain demands are unable to account for this sort of sharp persistent change in the market for a product.

**New products:** The problems facing managers of inventories of new products are, in a sense, the reverse of those arising from potential obsolescence: Demand is now just beginning. If the product is successful, demand will soon increase rapidly. The size and timing of the increase, however, cannot be predicted accurately. Again, the difficulty here is demand uncertainty of a very special kind, which standard models ignore.

So, in this paper, we assume that the environmental process follows a discrete-time Markov chain, introduce the model with a single-product inventory system of which demand distribution depends on environmental fluctuations, and discuss the management policy. The main advantage of the Markov-chain approach is that it provides a natural and flexible framework for formulating various changes described above.

Inventory models operating in random environments are only scarcely considered in earlier papers. For example, Kalymon [8] considers a discrete-time inventory purchasing model, in which the unit cost of the item is determined by a Markov process, and the distribution of demand in each period depends on the current cost. Feldman [5] models the demand environment as a continuous-time Markov chain. Given the state of environment, the demand forms a compound-Poisson process. But he studies only the stationary distribution of the inventory position. Song and Zipkin [14] derive some basic characteristics of optimal policies and develop algorithms for computing them in a continuous-review inventory model where the demand process is a Markov modulated Poisson process. Ozekici and Parlar [11] consider infinite-horizon periodic-review inventory models with unreliable suppliers where the demand, supply and cost parameters change with respect to a randomly changing environment.

The effect of a rondomly changing environment in other stochastic models in operations research is discussed in following papers. Cinlar and Ozekici [1] studied a model in reliability and maintenance where the failure rates of the components of a device depend on a semi-Markov environment process.

Eisen and Tainiter [4], Neuts [9], and Prabhu and Zhu [12] introduced a model where the arrival and service rates depend on a randomly changing environment.

The purpose of this paper is to show the existence of the environmental-dependent optimal (s, S) policy by analyzing finite- and infinite-horizon periodic-review inventory model where the demand distribution and cost parameters depend on a Markov environment process.

Under an (s, S) policy, an order is placed to increase the item's inventory level to the level S as soon as this inventory level reaches or drops below the level s. So, S and s are the order-up-to point and the reorder-point, respectively. In this paper, in particular, we focus on the finite-horizon analysis, since it gives us concrete and realistic insights.

This paper is organized as follows: Section 2 presents the formulation of the general problem as a dynamic programming model. Section 3, 4, and 5 provide analyses for the single-period, finite-horizon, and infinite-horizon problems, respectively. The paper concludes with some final remarks in Section 6.

## 2. Assumption and Notation

Consider a single-product periodic-review inventory system for N-periods.Let the period be numbered such that the final period is denoted as period 1, while the first period is denoted as period $N$.

The state of the environment observed at the beginning of period $n$ ($n = 1, 2, ..., N$) is represented by $I_n$ and we assume that $I = \{I_n; n \geq 0\}$ is Markov chain on a countable state space $E$ with a given transition matrix $P = P(i, j) = P[I_{n+1} = j | I_n = i]$. Let $X_n$ denote the inventory level observed at the beginning of period $n$. The basic assumption of this model is that the demand distribution and the cost-parameters at any period depend on the state of the environment at the beginning of that period. Therefore, the decision maker observes both the inventory level and the environment state to decide on the optimal order quantity which is delivered immediately.

If $D_n$ is the total demand during period $n$, then the demand process $D = \{D_n; n \geq 0\}$ is depend on the Markov chain $I$ so that its conditional distribution function is $A_i(z_n) = P[D_n \leq z_n | I_n = i]$, with the probability density function $a_i(z_n)$. Also, we assume $A_i(0) = 0$, $a_i(\cdot) > 0$.

We consider the following four types of costs: if the environmental state is $i$, a fixed ordering cost $K_i$ independent of the order quantity, a unit ordering cost $c_i$, a unit holding cost $h_i$ incurred at the end of period, and a unit shortage cost $p_i$ incurred at the end of period. To motivate ordering, we assume that $p_i > c_i$ as in standard models. Also, we assume that unsatisfied demands are fully backlogged.

Let $Y_n(i, x_n)$ be the order-up-to level if the environment is $i$ and the inventory level is $x_n$ at the beginning of period $n$. The admissibility condition requires that $Y_n(i, x_n) \geq x_n$ since we do not allow for discarding of any inventory without satisfying demand. It is noted that, for any $y_n$, the inventory level $X_n$ is a Markov chain, where

$$X_{n-1} = x_n + [y_n(i, x_n) - x_n]^+ - D_n$$

for $n \geq 0$. Figure 1 illustrates the behavior of the inventory level.

Now, let $V_i^n(x_n)$ be the minimum expected total discount cost of operating for $n$-period with the state of the environment $i$ and the initial inventory level $x_n$, under the best ordering decision is used at period $n$ through period 1. Then, a dynamic programming equation (DPE) for the problem can be given as

$$V_i^0(x_0) \equiv 0,$$

and

$$V_i^n(x_n) = \min_{y_n \geq x_n} \{K_i \delta(y_n - x_n) + G_i^n(y_n) - c_i x_n\}, \; n > 0, \; i \in E, \tag{1}$$

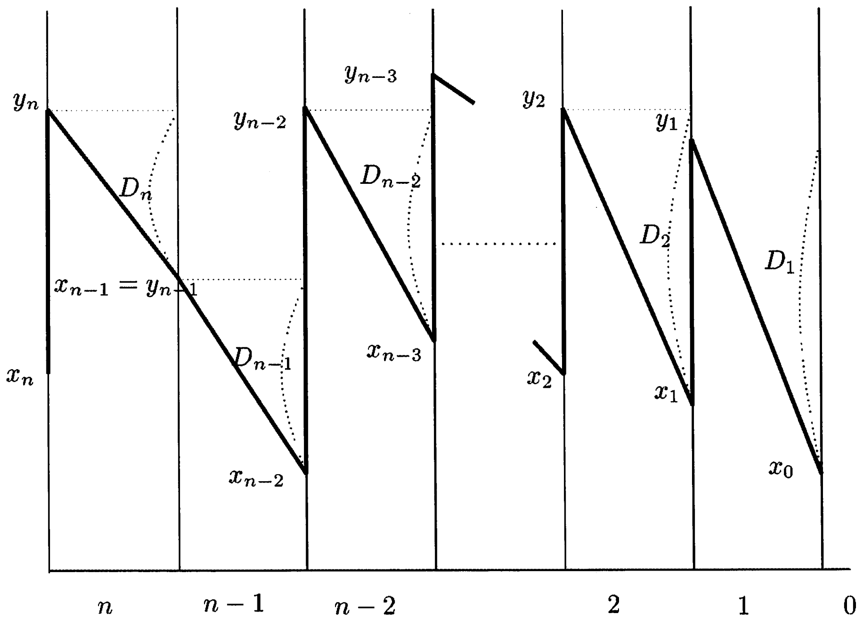


Figure 1: The behavior of the inventory level

where

$$\delta(y_n - x_n) = \begin{cases} 1 & \text{if} \quad y_n - x_n > 0, \\ 0 & \text{if} \quad y_n - x_n = 0, \end{cases}$$

and

$$G_i^n(y_n) = c_i y_n + L_i^n(y_n) + \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j^{n-1}(y_n - z_n) dA_i(z_n),$$

$$n > 0,\ i \in E \qquad (2)$$

with the expected holding and shortage cost function at period $n$

$$L_i^n(y_n) = h_i \int_0^{y_n} (y_n - z_n) dA_i(z_n) + p_i \int_{y_n}^\infty (z_n - y_n) dA_i(z_n)$$

and the discount factor $\alpha$ per period.

The decision variable in this model is $y_n$, so (2) plays a central role to find the optimal value $y_n^*$.

We assume that all parameters and costs are nonnegative, and that all relevant functions are differentiable.

## 3. Single-Period Analysis

In this section we analyze the single-period problem for the model introduced in the last section. This analysis will provide important insights in understanding the two-period analysis and $n$-period analysis. We begin by rewriting (1) and (2) as

$$V_i^1(x_1) = \min_{y_1 \geq x_1} \{K_i\delta(y_1 - x_1) + G_i^1(y_1) - c_i x_1\}, \tag{3}$$

$$G_i^1(y_1) = c_i y_1 + L_i^1(y_1), \tag{4}$$

where

$$L_i^1(y_1) = h_i \int_0^{y_1} (y_1 - z_1) dA_i(z_1) + p_i \int_{y_1}^{\infty} (z_1 - y_1) dA_i(z_1)$$

We first investigate the properties of (4) since it plays a central role in the minimization in (3). We obtain the first two derivatives of (4) as follows:

$$\frac{dG_i^1(y_1)}{dy_1} = G'^1_i(y_1) = c_i + (h_i + p_i)A_i(y_1) - p_i, \tag{5}$$

$$\frac{d^2 G_i^1(y_1)}{dy_1^2} = G''^1_i(y_1) = (h_i + p_i)a_i(y_1). \tag{6}$$

Then,

$$G''^1_i(y_1) = (h_i + p_i)a_i(y_1) > 0, \tag{7}$$

because $h_i$, $p_i$ and $a_i(y_1) > 0$. So, $G_i^1(y_1)$ is convex in $y_1$. It should be noted that the rightside of (5) is increasing in $y_1$,

$$\lim_{y_1 \to \infty} G'^1_i(y_1) = c_i + h_i > 0 \tag{8}$$

and

$$\lim_{y_1 \to 0} G'^1_i(y_1) = c_i - p_i < 0. \tag{9}$$

Therefore, there exists a unique solution such that

$$G'^1_i(y_1) = c_i + (h_i + p_i)A_i(y_1) - p_i = 0. \tag{10}$$

Let $S_i^1$ solve (10), that is,

$$S_i^1 = A_i^{-1}\left[\frac{p_i - c_i}{h_i + p_i}\right].$$

$S_i^1$ is nonnegative and finite because $0 < \left[\frac{p_i - c_i}{h_i + p_i}\right] < 1$ with $(p_i - c_i) > 0$ and $(p_i - c_i) < (h_i + p_i)$.

Now the property of (4) can be characterized by using (5) and (6):

(1) For $y_1 \leq S_i^1$, $G'^1_i(y_1) \leq 0$, $G''^1_i(y_1) > 0$, hence $G_i^1(y_1)$ is decreasing and convex.

(2) For $y_1 > S_i^1$ , $G'^1_i(y_1) > 0$ , $G''^1_i(y_1) > 0$ , hence $G_i^1(y_1)$ is increasing and convex.

From these observations, it is clear that $G_i^1(y_1)$ attains its global minimum at $y_1 = S_i^1$ with value $G_i^1(S_i^1)$ .

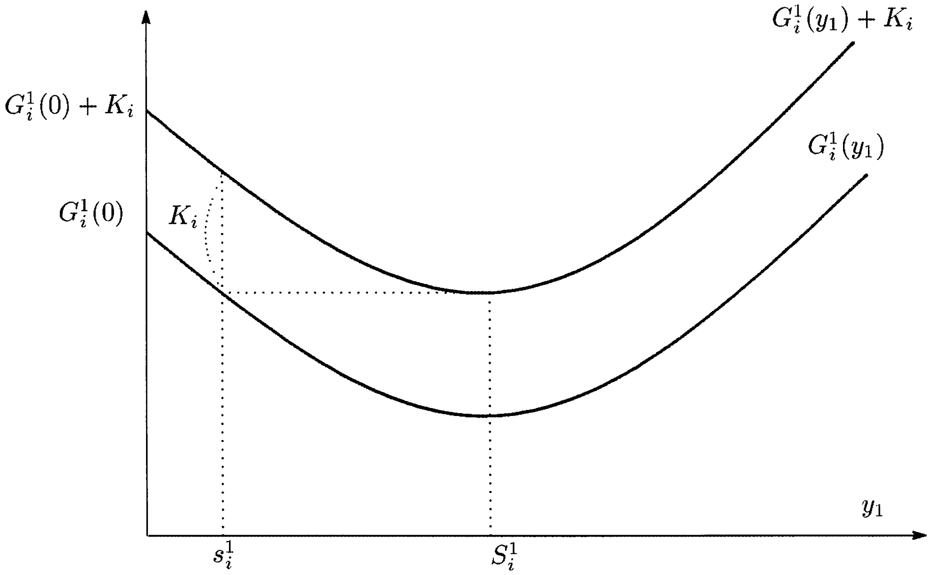


Figure 2: The form of $G_i^1(y_1)$ and $K_i + G_i^1(y_1)$

Now, consider the minimization in (3), in particular the term $(K_i\delta(y_1 - x_1) + G_i^1(y_1))$ . The nature of $(K_i + G_i^1(y_1))$ is identical to that of $G_i^1(y_1)$ and it attains the global minimum at $S_i^1$ with value $K_i + G_i^1(S_i^1)$ . For $y_1 \le S_i^1$ , since $G_i^1(y_1)$ is decreasing in $y_1$, there exists a unique solution such that

$$K_i + G_i^1(S_i^1) = G_i^1(y_1) \tag{11}$$

Let $s_i^1$ solve (11), then it follows from the definition of $s_i^1$ and the decreasing property of $G_i^1(y_1)$ for $y_1 \le S_i^1$ that

$$K_i + G_i^1(S_i^1) \le G_i^1(y_1) \text{ for } y_1 \le s_i^1 , \tag{12}$$

and

$$K_i + G_i^1(S_i^1) > G_i^1(y_1) \text{ for } s_i^1 < y_1 \le S_i^1 . \tag{13}$$

The behavior of $G_i^1(y_1)$ and $G_i^1(y_1) + K_i$ is shown graphically in Figure 2.

Based upon the state of the environment $i$ and the initial inventory level $x_1$, the optimal policy can now be characterized in terms of the two critical numbers $S_i^1$ and $s_i^1$. For $x_1 \le s_i^1$, the advantage $(G_i^1(x_1) - G_i^1(S_i^1))$ gained by ordering up to $S_i^1$ can offset the fixed ordering cost $K_i$ provided one plans to order. This follows from (12). On the other hand, for $s_i^1 < x_1 \le S_i^1$, it is not worthwhile to order because the fixed ordering cost $K_i$ will offset the expected savings $(G_i^1(x_1) - G_i^1(S_i^1))$ derived from ordering $(S_i^1 - x_1)$ units. This follows from (13). Since ordering will increase the expected cost for $x_1 > S_i^1$, it is not worthwhile to order, too.

Now the following summarizes the optimal policy for period 1 and the property of $G_i^1(y_1)$.

(1) the optimal policy for period 1 is given by

$$Y_1^*(i, x_1) = \begin{cases} S_i^1 \text{ if } x_1 \le s_i^1, \\ x_1 \text{ if } x_1 > s_i^1, \end{cases}$$

where critical numbers $S_i^1$ and $s_i^1$ are solutions to Eqs. (10) and (11) and are the order-up-to point and the reorder point, respectively.

(2) $G_i^1(y_1)$ is convex and $\begin{cases} \text{decreasing in } y_1 \text{ if } y_1 \le S_i^1, \\ \text{increasing in } y_1 \text{ if } y_1 > S_i^1, \end{cases}$

Therefore, the expected cost $V_i^1(x_1)$ under the optimal policy is obtained by substituting $Y_1^*(i, x_1)$ into (3):

$$V_i^1(x_1) = \begin{cases} K_i + G_i^1(S_i^1) - c_i x_1 & \text{if } x_1 \le s_i^1, \\ G_i^1(x_1) - c_i x_1 & \text{if } x_1 > s_i^1. \end{cases} \tag{14}$$

And its first two derivatives are

$$V'^1_i(x_1) = \begin{cases} -c_i & \text{if } x_1 \le s_i^1, \\ G'^1_i(x_1) - c_i & \text{if } x_1 > s_i^1, \end{cases} \tag{15}$$

$$V''^1_i(x_1) = \begin{cases} 0 & \text{if } x_1 \le s_i^1, \\ G''^1_i(x_1) & \text{if } x_1 > s_i^1. \end{cases} \tag{16}$$

So, $V_i^1(x_1)$ is quasi-convex in $x_1$.

## 4. n-Period Analysis

In this section, we analyze the n-period problem for the model introduced in Section 2.

To use induction, we assume that the following properties hold for the (n-1)-period problem, where the state of the environment is $j \in E$.

$$Y^*_{n-1}(j, x_{n-1}) = \begin{cases} S_j^{n-1} & \text{if } x_{n-1} \le s_j^{n-1}, \\ x_{n-1} & \text{if } x_{n-1} > s_j^{n-1}, \end{cases} \tag{17}$$

$$G'^{n-1}_j(S_j^{n-1}) = 0 \tag{18}$$

$$K_i + G_i^{n-1}(S_i^{n-1}) = G_i^{n-1}(s_i^{n-1}) \tag{19}$$

$$G''^{n-1}_j(\cdot) > 0 \tag{20}$$

$$\lim_{y_{n-1} \to \infty} G'^{n-1}_j(y_{n-1}) = c_j + h_j + \alpha \sum_{k \in E} P(j,k)h_k + \alpha^2 \sum_{k,l \in E} P(j,k)P(k,l)h_l + \cdots + \alpha^{n-2} \sum P(j,k)P(k,l)\cdots P(\chi, \psi)P(\psi, \omega)h_\omega > 0 \tag{21}$$

$$\lim_{y_{n-1} \to 0} G'^{n-1}_j(y_{n-1}) < c_j - \alpha \sum_{k \in E} P(j,k)c_k - p_j < 0 \tag{22}$$

$$V_j^{n-1}(x_{n-1}) = \begin{cases} K_j + G_j^{n-1}(S_j^{n-1}) - c_j x_{n-1} & \text{if } x_{n-1} \le s_j^{n-1}, \\ G_j^{n-1}(x_{n-1}) - c_j x_{n-1} & \text{if } x_{n-1} > s_j^{n-1}. \end{cases} \tag{23}$$

$$V'^{n-1}_j(x_{n-1}) = \begin{cases} -c_j & \text{if } x_{n-1} \le s_j^{n-1}, \\ G'^{n-1}_j(x_{n-1}) - c_j & \text{if } x_{n-1} > s_j^{n-1}. \end{cases} \tag{24}$$

$$V''^{n-1}_j(x_{n-1}) = \begin{cases} 0 & \text{if } x_{n-1} \le s_j^{n-1}, \\ G''^{n-1}_j(x_{n-1}) & \text{if } x_{n-1} > s_j^{n-1}. \end{cases} \tag{25}$$

For an n-period problem, the DPE is given by (1). We investigate the property of (2) since it plays a central role in the minimization in (1).

First, we rewrite (2) by substituting $V_j^{n-1}$ from (23) into it as

$$\begin{aligned} G_i^n(y_n) &= y_n\{c_i - \alpha \sum_{j \in E} P(i,j)c_j\} + L_i^n(y_n) \\ &\quad + \alpha \sum_{j \in E} P(i,j)\Big[\int_{y_n - s_j^{n-1}}^{\infty} \{K_j + G_j^{n-1}(S_j^{n-1}) - c_j z_n\}\, dAi(z_n) \\ &\quad + \int_0^{y_n - s_j^{n-1}} \{G_j^{n-1}(y_n - z_n) - c_j z_n\}\, dA_i(z_n)\Big]. \end{aligned} \tag{26}$$

And, from (24), (25), we obtain the first and the second order derivatives for (26) as follows:

$$G'^{n}_{i}(y_n) = c_i - \alpha \sum_{j \in E} P(i,j)c_j + (h_i + p_i)A_i(y_n) - p_i + \alpha \sum_{j \in E} P(i,j)$$
$$\times \int_0^{y_n - s_j^{n-1}} G'^{n-1}_{j}(y_n - z_n)dA_i(z_n). \tag{27}$$

$$G''^{n}_{i}(y_n) = (h_i + p_i)a_i(y_n) + \alpha \sum_{j \in E} P(i,j)\int_0^{y_n - s_j^{n-1}} G''^{n-1}_{j}(y_n - z_n)dA_i(z_n). \tag{28}$$

Then,

$$G''^{n}_{i}(y_n) = (h_i + p_i)a_i(y_n) + \alpha \sum_{j \in E} P(i,j)\int_0^{y_n - s_j^{n-1}} G''^{n-1}_{j}(y_n - z_n)dA_i(z_n) > 0, \tag{29}$$

because $h_i$, $p_i$, $a_i(y_n)$ and $G''^{n-1}_{j}(\cdot) > 0$. So, $G^n_i(y_n)$ is convex in $y_n$. The rightside of (27) is increasing in $y_n$,

$$\lim_{y_n \to \infty} G'^{n}_{i}(y_n) = c_i + h_i + \alpha \sum_{j \in E} P(i,j)h_j + \alpha^2 \sum_{j,k \in E} P(i,j)P(j,k)h_k$$
$$+\alpha^3 \sum_{j,k,l \in E} P(i,j)P(j,k)P(k,l) + \cdots + \alpha^{n-1} \sum P(i,j)$$
$$\times P(j,k)P(k,l) \cdots P(\chi,\psi)P(\psi,\omega)h_\omega > 0, \tag{30}$$

and

$$\lim_{y_n \to 0} G'^{n}_{i}(y_n) = c_i - \alpha \sum_{j \in E} P(i,j)c_j - p_i + \alpha \sum_{j \in E} P(i,j)\int_0^{-s_j^{n-1}} G'^{n-1}_{j}(-z_n)$$
$$dA_i(z_n) < c_i - \alpha \sum_{j \in E} P(i,j)c_i - p_i < 0. \tag{31}$$

Therefore, there exists a unique solution such that

$$G'^{n}_{i}(y_n) = c_i - \alpha \sum_{j \in E} P(i,j)c_j + (h_i + p_i)A_i(y_n) - p_i$$
$$+\alpha \sum_{j \in E} P(i,j)\int_0^{y_n - s_j^{n-1}} G'^{n-1}_{j}(y_n - z_n)dA_i(z_n) = 0. \tag{32}$$

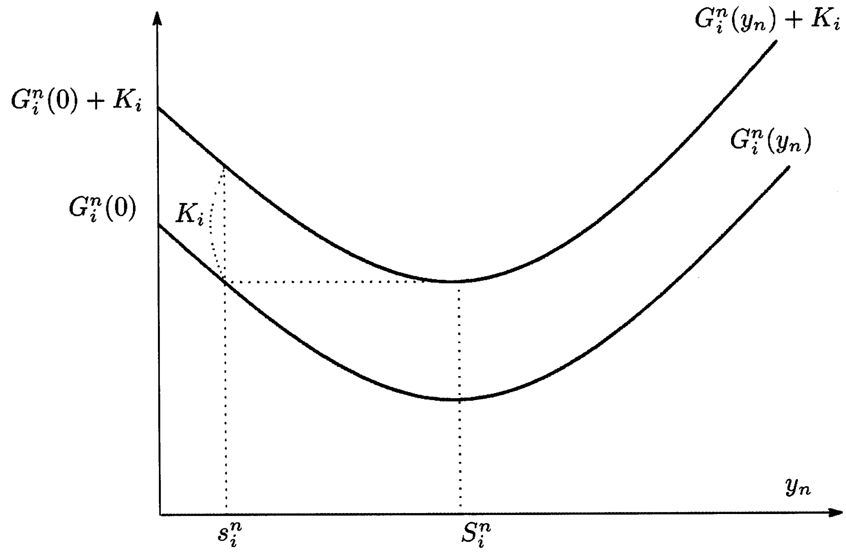


Figure 3: The form of $G_i^n(y_n)$ and $K_i + G_i^n(y_n)$

Let $S_i^n$ solve (32), then the property of (2) is as follows:
(1) For $y_n \le S_i^n$, $G'^n_i(y_n) \le 0$, $G''^n_i(y_n) > 0$ , hence $G_i^n(y_n)$ is decreasing and convex.
(2) For $y_n > S_i^n$, $G'^n_i(y_n) > 0$, $G''^n_i(y_n) > 0$ , hence $G_i^n(y_n)$ is increasing and convex.

So, $G_i^1(y_1)$ attains its global minimum at $y_n$=$S_i^n$ with value $G_i^n(S_i^n)$ .

Now, consider the minimization in (1), in particular the term $(K_i\delta(y_n - x_n) + G_i^n(y_n))$ . The nature of $(K_i + G_i^n(y_n))$ is identical to that of $G_i^n(y_n)$ and it attains the global minimum at $S_i^n$ with value $K_i + G_i^n(S_i^n)$ . For $y_n \le S_i^n$ , since $G_i^n(y_n)$ is decreasing in $y_n$, there exists a unique solution such that

$$K_i + G_i^n(S_i^n) = G_i^n(y_n) . \tag{33}$$

Let $s_i^n$ solve (33), then it follows from the definition of $s_i^n$ and the decreasing property of $G_i^n(y_n)$ for $y_n \le S_i^n$ that

$$K_i + G_i^n(S_i^n) \le G_i^n(y_n) \text{ for } y_n \le s_i^n, \tag{34}$$

$$K_i + G_i^n(S_i^n) > G_i^n(y_n) \text{ for } s_i^n < y_n \le S_i^n. \tag{35}$$

The behavior of $G_i^n(y_n)$ and $G_i^n(y_n) + K_i$ is shown graphically in Figure 3.

So, based upon the state of the environment $i$ and the initial inventory level $x_n$, the optimal policy can now be characterized in terms of the two critical numbers $S_i^n$ and $s_i^n$ . For $x_n \le s_i^n$ , the advantage $(G_i^n(x_n) - G_i^n(S_i^n))$ gained by ordering up to $S_i^n$ can offset the fixed ordering cost $K_i$ provided one plans to order. This follows from (34). On the other hand, for $s_i^n < x_n \le S_i^n$ , it is not worthwhile to order because the fixed ordering cost $K_i$ will offset the expected savings $(G_i^n(x_n) - G_i^n(S_i^n))$ derived from ordering $(S_i^n - x_n)$ units. This follows from (35). Since the ordering will increase the expected cost for $x_n > S_i^n$ , it is not worthwhile to order, too.

Now the following summarizes the optimal policy for period n and the property of $G_i^n(y_n)$ .

(1) the optimal policy for period $n$ is

$$Y_n^*(i, x_n) = \begin{cases} S_i^n & \text{if } x_n \le s_i^n, \\ \chi_n & \text{if } x_n > s_i^n, \end{cases}$$

where critical numbers $S_i^n$ and $s_i^n$ are solutions to Eqs. (32) and (33) and are the order-up-to point and the reorder point, respectively.

(2) $G_i^n(y_n)$ is convex and $\begin{cases} \text{decreasing in } y_n & \text{if } y_n \le S_i^n, \\ \text{increasing in } y_n & \text{if } y_n > S_i^n. \end{cases}$

Therefore, the expected cost $V_i^n(x_n)$ under the optimal policy is obtained by substituting $Y_n^*(i, x_n)$ into (1):

$$V_i^n(x_n) = \begin{cases} K_i + G_i^n(S_i^n) - c_i x_n & \text{if } x_n \le s_i^n, \\ G_i^n(x_n) - c_i x_n & \text{if } x_n > s_i^n. \end{cases} \tag{36}$$

And its first two derivatives are given by

$$V'^n_i(x_n) = \begin{cases} -c_i & \text{if } x_n \le s_i^n, \\ G'^n_i(x_n) - c_i & \text{if } x_n > s_i^n, \end{cases} \tag{37}$$

$$V''^n_i(x_n) = \begin{cases} 0 & \text{if } x_n \le s_i^n, \\ G''^n_i(x_n) & \text{if } x_n > s_i^n. \end{cases} \tag{38}$$

So, $V_i^n(x_n)$ is quasi-convex in $x_n$.

## 5. Infinite-Horizon Analysis

In this section, we consider the case where $n \to \infty$ for the model introduced in section 1. For an infinite-horizon problem, the DPE, which is the equivalent to (1), can be written as

$$V_i(x) = \min_{y \ge x} \{K_i \delta(y - x) + G_i(y) - c_i x\}, \tag{39}$$

where

$$G_i(y) = c_i y + L_i(y) + \alpha \sum_{j \in E} P(i, j) \int_0^\infty V_j(y - z) dA_i(z). \tag{40}$$

Our purpose in this section is to show that the DPE $V_i^n(x)$ of the finite-horizon problem converges to a limit function $V_i(x)$, which satisfies (39), (40), and that the order-up-to point $S_i^n$ and the reorder point $s_i^n$ of the finite-horizon problem also converge to $S_i$, $s_i$, respectively, where $S_i$ and $s_i$ specify the optimal ordering policy for (39).

The following gives the proof.

Consider the convergence of the functional sequence $\{V_i^n(x)\}_{n=1}^\infty$ defined as

$$V_i^1(x) = \min_{y \geq x}\{K_i\delta(y-x) + c_i(y-x) + L_i(y)\},$$

$$V_i^n(x) = \min_{y \geq x}\{K_i\delta(y-x) + c_i(y-x) + L_i(y)$$

$$+\alpha \sum_{j \in E} P(i,j)\int_0^{\infty} V_j^{n-1}(y-z)dA_i(z)\} \text{ . } n = 2, 3, \cdots \tag{40}$$

Note that $V_i^1(x)$ is continuous in $x$ from (14) and so is $V_i^n(x)$ for each $n$ recursively from (41). And, for any $x$,

$$V_i^2(x) \geq \min_{y \geq x}\{K_i\delta(y-x) + c_i(y-x) + L_i(y)\} = V_i^1(x) \geq 0,$$

therefore, $\{V_i^n(x)\}_{n=1}^{\infty}$ is nondecreasing recursively from (41), i.e.,

$$0 \leq V_i^1(x) \leq V_i^2(x) \leq \cdots\cdots \leq V_i^n(x) \leq \cdots.$$

Since $\{V_i^n(x) - V_i^{n-1}(x)\} \geq 0$ is also continuous in $x$ for sufficiently large $X > 0$, there exists a maximum value in the closed interval $-X \leq x \leq X$ from the Weierstrass M test. Let the value be

$$u_i^n = \max_{-X \leq x \leq X}\{V_i^n(x) - V_i^{n-1}(x)\} \text{ , } n = 2, 3 \cdots,$$

then, since

$$\left|V_i^{n+1}(x) - V_i^n(x)\right| \leq \alpha \sum_{j \in E} P(i,j)\min_{y \geq x}\left\{\int_0^{\infty}\left|V_j^n(y-z) - V_j^{n-1}(y-z)\right|dA_i(z)\right\},$$

$$u_i^{n+1} \leq \alpha \sum_{j \in E} P(i,j)u_j^n, \; n = 1, 2, \cdots.$$

Hence,

$$u_i^n \leq \alpha \sum_{j_1 \in E} P(i,j_1)u_{j_1}^{n-1} \leq \alpha^2 \sum_{j_1, j_2 \in E} P(i,j_1)P(j_1,j_2)u_{j_2}^{n-2} \leq \cdots$$

$$\ldots \leq \alpha^{n-1} \sum_{j_1, \cdots, j_{n-1} \in E} P(i,j_1)P(j_1,j_2)\cdots P(j_{n-2}, j_{n-1})u_{j_{n-1}}^1, \; n = 2, 3, \cdots.$$

Since $\sum_{n=2}^{\infty} u_i^n$ is convergent, the positive term series $\sum_{n=2}^{\infty}\{V_i^{n+1}(x) - V_i^n(x)\}$ is uniformly convergent for $x \leq |X|$ .

Therefore, because

$$V_i^n(x) = V_i^1(x) + \{V_i^2(x) - V_i^1(x)\} + \cdots\cdots + \{V_i^n(x) - V_i^{n-1}(x)\} \text{ ,}$$

the functional sequence $\{V_i^n(x)\}_{n=1}^{\infty}$ is uniformly convergent for $x \leq |X|$ . Let the limitting function be $V_i(x)$, then $V_i(x)$ is continuous because of its uniform convergency, and

$$V_i(x) = \min_{y \geq x}\{K_i\delta(y-x) + G_i(y) - c_ix\},$$

where

$$G_i(y) = c_i y + L_i(y) + \alpha \sum_{j \in E} P(i,j) \int_0^{\infty} V_j(y-z)\, dA_i(z) .$$

Hence,

$$V_i(x) = \lim_{n \to \infty} V_i^n(x) ,$$

$$G_i(y) = \lim_{n \to \infty} G_i^n(y) .$$

So, $G_i(y)$ attains the global minimum at the point such that $G'_i(y) = 0$ .
Let $S_i$ solve $\{ G'_i(y) = 0 \}$ , then, since the functional sequence which is uniformly convergent and continuous is partial differentiable with term by term,

$$G'_i(y) = \lim_{n \to \infty} G'^n_i(y) .$$

Hence,

$$S_i = \lim_{n \to \infty} S_i^n ,$$

and let solve $K_i + G_i(y) = G_i(S_i)$ , then

$$s_i = \lim_{n \to \infty} s_i^n .$$

## 6. Concluding Remarks

In this paper, first, we show the existence of the environment-dependent optimal (s, S) policy by analyzing finite-horizon periodic-review inventory model where the demand distribution depends on a Markov environment process. We further show that the solution for the finite-horizon problem converges to that of the infinite-horizon problem.

We discuss the limitation of our model as well as possible extensions.

**Nonstationary transition probability:** In this paper, the probability of environment changing is decreasing in time (periods), since we assume that the environment is Markov chain. But, there is a case that it is increasing in time, like the obsolescence of products. To solve the contradiction, we must introduce a nonstastionary transition probability.

**Another policy:** In our model, we consider the optimal policy that let the order-up-to point be the decision variable. So, we next time consider the optimal policy that let the order quantity, the reorder-point, and the order intervals be the decision variable. Thereby, we decide the real optimal policy by comparing them.

**Endogenous factors:** In this paper, we analyze the inventory model that depends on the exogenous factors. So, we present the inventory model that depends on the endogenous factors where the demand be influenced by the order quantity for example.

**Another uncertain element:** We will introduce to our model the unreliable suppliers or the uncertain leadtime, since these are depend on the environment.

**Production and distribution:** Recently, like KANBAN of TOYOTA, zero-inventory policy becomes main topic. So, hereafter, we focus on the inventory management to minimize the cost in the system combined with production or distribution.

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