Exercise Strategies for the Bermudan Swaption

Yasuhiro Tamba

Abstract

This paper presents theoretical conditions, under which an option holder does not exercise a Bermudan swaption. We can utilize the conditions for making profitable exercise strategies. The conditions are derived by optimality equations under varying forward neutral probabilities.

Keywords: Bermudan swaption, dynamic programming, risk neutral evaluation, exercise strategy.

1 Introduction

Exotic interest rate derivatives are flexible financial instruments which satisfy demands for hedging interest rate risk in a financial market. One of the most traded exotic interest rate derivatives is a Bermudan swaption. The Bermudan swaption is an option, which at each date in a schedule of exercise dates gives the holder the right to enter an interest swap, provided that this right has not been exercised at any previous time in the schedule. Because of its usefulness as hedges for callable bonds, the Bermudan swaption is probably the most liquid interest rate instrument with a built-in early exercise feature. Its trade volume has increased for recent years in the market.

There are many papers for pricing the Bermudan swaption because of its popularity in the market. The pricing method used in most papers is a Monte Carlo simulation. Improved Monte Carlo methods for pricing the Bermudan swaption have been proposed by many researchers like Longstaff and Schwartz (1998) and Andersen (1999). Broadie and Glasserman (1997a, 1997b) developed the stochastic mesh method. Carr and Yang (1997) developed a method based on the stratification technique. But there is no research discussing Bermudan swaption's properties which can be utilized for profitable exercise strategies. In this paper we derive theoretical conditions, under which the option holder does not exercise the Bermudan swaption. The conditions are derived by optimality equations under varying forward neutral probabilities, which have not been used in past researches. We can utilize this property for making profitable exercise strategies.

The paper is organized as follows. In Section 2, we introduce various notations of interest rates. In Section 3 we derive theoretical conditions, under which the option holder does not exercise the Bermudan swaption at the terminal period. In Section 4 we derive theoretical conditions, under which the option holder does not exercise the Bermudan swaption at previous periods. Section 5 concludes the paper.

2 Notations of Interest Rates

Let $D(t,T)$ $0 \leq t \leq T \leq T^*$ be the time t price of the discount bond (or zero–coupon bond) with maturity T, in brief T–bond, which pays 1–unit of money at the maturity T (where $D(T,T)=1$ for any $T \in \mathbb{T}^*$). For $N \in \mathbb{Z}_+$, let

$$0 \leq T_0 < T_1 < \cdots < T_i < T_{i+1} < \cdots < T_{N-1} < T_N \leq T^* \tag{1}$$

be the sequence of setting times and payment times of floating interest rates, that is, for $i = 0, \cdots, N-1$, the floating interest rate which covers time interval $(T_i, T_{i+1}]$, is set at time T_i and paid at time T_{i+1}. For convenience, we let

$$T_{i+1} - T_i = \delta \ (= \text{constant} \in \mathbb{R}_{++}), \quad i = 0, \cdots, N-1. \tag{2}$$

For $i = 0, \cdots, N-1$, we define the simple (or simple compounding based) interest rate which covers time interval $(T_i, T_{i+1}]$ by

$$L_{T_i}(T_i) := \frac{1}{\delta}\left\{\frac{1}{D(T_i, T_{i+1})} - 1\right\}. \tag{3}$$

This amount is set at time T_i, paid at time T_{i+1}, and is conventionally called as a spot LIBOR (London Inter–Bank Offer Rate). For $i = 0, \cdots, N-1$,

$$L_{T_i}(t) := \frac{1}{\delta}\left\{\frac{D(t, T_i)}{D(t, T_{i+1})} - 1\right\} \tag{4}$$

is the simple (or simple compounding based) interest rate prevailing at time t ($\in [0, T_i]$) which covers time interval $(T_i, T_{i+1}]$, and is called as a forward LIBOR.

An interest rate swap is a contract where two parties agree to exchange a set of floating interest rate, LIBOR, payments for a set of fixed interest rate payments. In the market, swaps are not quoted as prices for different fixed rates K, but only the fixed rate K is quoted for each swap such that the present value of the swap is equal to zero. This rate, called the par swap rate $S(t)$ at t, with the payments from T_1 to T_n is calculated as

$$S(t) = \frac{D(t, T_0) - D(t, T_n)}{\sum_{k=1}^{n} \delta D(t, T_k)}. \tag{5}$$

3 Conditions for Non Early Exercise of the Bermudan Swaption at $t = T_{N-2}$

In this section we derive theoretical conditions, under which the option holder does not exercise the Bermudan swaption at the terminal period, T_{N-2}.

Proposition 3.1. *The holder of the Bermudan swaption does not exercise the Bermudan swaption at* $t = T_{N-2}$ *under the conditions*

$$S(T_{N-2}) < \frac{1}{1 + \delta L_{N+l-2}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}), l = 1, 2; \tag{6}$$

$$S(T_{N-2}) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{N+s-3}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}), l = 3, 4, \cdots, n. \quad (7)$$

Proof. The value of the Bermudan swaption at the terminal period, T_{N-1}, is

$$W(S(T_{N-1})) = \delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k}). \quad (8)$$

The optimality equation at T_{N-2} is

$$\begin{aligned} W(S(T_{N-2})) &= \max\Big\{\delta[S(T_{N-2}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-2}, T_{N+k-1}), \\ &\qquad D(T_{N-2}, T_N) E^{T_N}\Big[\frac{W(S(T_{N-1}))}{D(T_{N-1}, T_N)}\Big| S(T_{N-2})\Big]\Big\} \\ &= \max\Big\{\delta[S(T_{N-2}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-2}, T_{N+k-1}), \\ D(T_{N-2}, T_N) E^{T_N}&\Big[\frac{\delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k})}{D(T_{N-1}, T_N)}\Big| S(T_{N-2})\Big]\Big\}. \end{aligned} \quad (9)$$

The condition, under which the option holder does not exercises the Bermudan swaption at T_{N-2}, is

$$\begin{aligned} &\delta[S(T_{N-2}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-2}, T_{N+k-1}) \\ &< D(T_{N-2}, T_N) E^{T_N}\Big[\frac{\delta[S(T_{N-1}) - K]_+ \sum_{k=0}^{n-1} D(T_{N-1}, T_{N+k})}{D(T_{N-1}, T_N)}\Big| S(T_{N-2})\Big]\Big\}. \end{aligned} \quad (10)$$

Using an approximation,

$$S(T_{N-1}) \approx \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-1}), \quad (11)$$

where

$$w_i(t) = \frac{D(t, T_i)}{\sum_{k=0}^{n-1} D(t, T_{i+k})}, \quad (12)$$

we have the first term of the summation on the right hand side of (10), A, as

$$\begin{aligned} A &= D(T_{N-2}, T_N) E^{T_N}\Big[\delta[S(T_{N-1}) - K]_+\Big| S(T_{N-2})\Big] \\ &\approx D(T_{N-2}, T_N) E^{T_N}\Big[\delta[\sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-1}) - K]_+\Big| S(T_{N-2})\Big] \\ &\geq \delta D(T_{N-2}, T_N)[\sum_{k=0}^{n-1} w_{N+k}(0) E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big| S(T_{N-2})\Big] - K]_+. \end{aligned} \quad (13)$$

We evaluate the expectation terms of the equation (13). The first term of the summation is evaluated as

$$E^{T_N}\left[L_{N-1}(T_{N-1})\middle|S(T_{N-2})\right] = L_{N-1}(T_{N-2}). \tag{14}$$

Next we evaluate the second term of the summation, $E^{T_N}\left[L_N(T_{N-1})\middle|S(T_{N-2})\right]$. We consider the payoff $L_N(T_{N-1})$ given at T_N. We evaluate the payoff under each of the forward neutral probability measure of $\mathbb{P}^{T_N}$ and $\mathbb{P}^{T_{N+1}}$. We define P as the evaluated value at T_{N-2} corresponding to the payoff.

$$\frac{P}{D(T_{N-2},T_N)} = E^{T_N}\left[\frac{L_N(T_{N-1})}{D(T_N,T_N)}\middle|S(T_{N-2})\right] \tag{15}$$

$$\frac{P}{D(T_{N-2},T_{N+1})} = E^{T_{N+1}}\left[\frac{L_N(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right] \tag{16}$$

So we have

$$E^{T_N}\left[L_N(T_{N-1})\middle|S(T_{N-2})\right] = \frac{D(T_{N-2},T_{N+1})}{D(T_{N-2},T_N)}E^{T_{N+1}}\left[\frac{L_N(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right]. \tag{17}$$

Utilizing the facts that

$$D(T_N,T_{N+1}) = \frac{1}{1+\delta L_N(T_N)} \tag{18}$$

and the function

$$f(x) := x(1+\delta x) \tag{19}$$

is convex in x, we evaluate the expectation term in the equation (17) as

$$\begin{aligned} E^{T_{N+1}}\left[\frac{L_N(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right] &= E^{T_{N+1}}\left[L_N(T_{N-1})\{1+\delta L_N(T_N)\}\middle|S(T_{N-2})\right] \\ &\geq L_N(T_{N-2})(1+\delta L_N(T_{N-2})). \end{aligned} \tag{20}$$

Hence we have the relationship

$$E^{T_N}\left[L_N(T_{N-1})\middle|S(T_{N-2})\right] \geq L_N(T_{N-2}). \tag{21}$$

Next we evaluate the third term of the summation, $E^{T_N}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right]$, in the same way. We consider the payoff $L_{N+1}(T_{N-1})$ at T_N. We evaluate the payoff under each of the forward neutral probability measure of $\mathbb{P}^{T_N}$ and $\mathbb{P}^{T_{N+1}}$. We define Q as the evaluated value at T_{N-2} corresponding to the payoff.

$$\frac{Q}{D(T_{N-2},T_N)} = E^{T_N}\left[\frac{L_{N+1}(T_{N-1})}{D(T_N,T_N)}\middle|S(T_{N-2})\right] \tag{22}$$

$$\frac{Q}{D(T_{N-2},T_{N+1})} = E^{T_{N+1}}\left[\frac{L_{N+1}(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right] \tag{23}$$

So we have

$$
\begin{aligned}
& E^{T_N}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right] \\
= & \tfrac{D(T_{N-2},T_{N+1})}{D(T_{N-2},T_N)} E^{T_{N+1}}\left[\tfrac{L_{N+1}(T_{N-1})}{D(T_N,T_{N+1})}\middle|S(T_{N-2})\right].
\end{aligned} \tag{24}
$$

Assuming that the following Brownian motions are uncorrelated

$$
\mathrm{d}W^{T_{N+2}}(t)\mathrm{d}W^{T_{N+1}}(t) = 0, \tag{25}
$$

where

$$
\frac{\mathrm{d}L_{N+1}(t)}{L_{N+1}(t)} = \sigma_{N+1}(t)\mathrm{d}W^{T_{N+2}}(t), \quad t \in \mathbb{T}^* \tag{26}
$$

$$
\frac{\mathrm{d}L_N(t)}{L_N(t)} = \sigma_N(t)\mathrm{d}W^{T_{N+1}}(t), \quad t \in \mathbb{T}^*, \tag{27}
$$

we have the expectation term in (24) as

$$
\begin{aligned}
& E^{T_{N+1}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_N,T_{N+1})}\Big|S(T_{N-2})\Big] \\
= \; & E^{T_{N+1}}\left[1+\delta L_N(T_N)\middle|S(T_{N-2})\right] E^{T_{N+1}}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right].
\end{aligned} \tag{28}
$$

We evaluate the term $E^{T_{N+1}}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right]$ in (28). We consider the payoff $L_{N+1}(T_{N-1})$ at T_{N+1}. We evaluate the payoff under each of the forward neutral probability measure of $\mathbb{P}^{T_N+1}$ and $\mathbb{P}^{T_{N+2}}$. We define R as the evaluated value at T_{N-2}.

$$
\frac{R}{D(T_{N-2},T_{N+1})} = E^{T_{N+1}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+1})}\Big|S(T_{N-2})\Big] \tag{29}
$$

$$
\frac{R}{D(T_{N-2},T_{N+2})} = E^{T_{N+2}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+2})}\Big|S(T_{N-2})\Big] \tag{30}
$$

So we have

$$
\begin{aligned}
& E^{T_{N+1}}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right] \\
= \; & \frac{D(T_{N-2},T_{N+2})}{D(T_{N-2},T_{N+1})} E^{T_{N+2}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+2})}\Big|S(T_{N-2})\Big].
\end{aligned} \tag{31}
$$

In the same way as (20), we have the expectation term in (31) as

$$
E^{T_{N+2}}\Big[\frac{L_{N+1}(T_{N-1})}{D(T_{N+1},T_{N+2})}\Big|S(T_{N-2})\Big] \geq L_{N+1}(T_{N-2})(1+\delta L_{N+1}(T_{N-2})). \tag{32}
$$

Hence we evaluate the third term of the summation as

$$
E^{T_N}\left[L_{N+1}(T_{N-1})\middle|S(T_{N-2})\right] \geq L_{N+1}(T_{N-2}). \tag{33}
$$

In the same way we have the relations

$$E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big|S(T_{N-2})\Big] \geq L_{N+k-1}(T_{N-2}),\ k=1,2,\cdots,n-1. \tag{34}$$

Then A has the following relation.

$$A \geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2})-K]_+ \tag{35}$$

Comparing the first term of the left side on the equation (10) and the right side of equation (35), we have

$$[S(T_{N-2})-K]_+ \leq \frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N-1})}[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2})-K]_+. \tag{36}$$

$$RHSof(36) \geq [\frac{1}{1+\delta L_{N-1}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2})-K]_+ \tag{37}$$

Hence one of the sufficient conditions for non early exercise of the Bermudan swaption at $t=T_{N-2}$ derived from the comparison of the first terms is

$$S(T_{N-2}) \leq \frac{1}{1+\delta L_{N-1}(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}). \tag{38}$$

We have the second term of the summation on the right hand side of (10), B, as

$$\begin{aligned}
B &= D(T_{N-2},T_N)E^{T_N}\Big[\frac{D(T_{N-1},T_{N+1})\delta[S(T_{N-1})-K]_+}{D(T_{N-1},T_N)}\Big|S(T_{N-2})\Big] \\
&\approx \delta D(T_{N-2},T_N)E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-1})-K]_+\Big|S(T_{N-2})\Big] \\
&\geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big]-K]_+. \qquad (39)
\end{aligned}$$

We evaluate the expectation terms of the equation (39). Under the following assumption that all Brownian motions are uncorrelated each other

$$\mathrm{d}W^{T_i}(t)\mathrm{d}W^{T_j}(t)=0, \quad i\neq j, \tag{40}$$

we have

$$\begin{aligned}
& E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big] \\
= \ & E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big]E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big|S(T_{N-2})\Big]. \qquad (41)
\end{aligned}$$

Because the function

$$g(x) := \frac{1}{1+\delta x} \tag{42}$$

is convex in x for $x \geq 0$, we have

(41)

Because the function

$$E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big] \geq \frac{1}{1+\delta L_N(T_{N-2})}. \tag{43}$$

Utilizing (43) and (34) we derive the relation

(42)
is convex in x for $x \geq 0$, we have

$$E^{T_N}\Big[\frac{1}{1+\delta L_N(T_{N-1})}\Big|S(T_{N-2})\Big]E^{T_N}\Big[L_{N+k-1}(T_{N-1})\Big|S(T_{N-2})\Big]$$
$$\geq \frac{L_{N+k-1}(T_{N-2})}{1+\delta L_N(T_{N-2})}. \tag{44}$$

(43)
Utilizing (43) and (34) we derive the relation

Hence B has the following relation.

(44)

$$B \geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)\frac{L_{N+k-1}(T_{N-2})}{1+\delta L_N(T_{N-2})} - K]_+. \tag{45}$$

Hence B has the following relation.

Comparing the second terms of the left side on the equation (10) and the right side of equation (45), we have

(45)
Comparing the second terms of the left side on the equation (10) and the right side of equation (45), we have

$$[S(T_{N-2}) - K]_+ \leq \frac{D(T_{N-2},T_N)}{D(T_{N-2},T_N)}[\sum_{k=0}^{n-1} w_{N+k}(0)\frac{L_{N+k-1}(T_{N-2})}{1+\delta L_N(T_{N-2})} - K]_+. \tag{46}$$

(46)

Hence one of the sufficient conditions for non early exercise of the Bermudan swaption at $t = T_{N-2}$ derived from the comparison of the second terms is

Hence one of the sufficient conditions for non early exercise of the Bermudan swaption at $t = T_{N-2}$ derived from the comparison of the second terms is

$$S(T_{N-2}) \leq \frac{1}{1+\delta L_N(T_{N-2})}\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-2}). \tag{47}$$

We have the third term of the summation on the right hand side of (10), C, as

$$C = D(T_{N-2},T_N)E^{T_N}\Big[\frac{D(T_{N-1},T_{N+2})\delta[S(T_{N-1})-K]_+}{D(T_{N-1},T_N)}\Big|S(T_{N-2})\Big]$$
$$\approx \delta D(T_{N-2},T_N)E^{T_N}\Big[\frac{1}{(1+\delta L_N(T_{N-1}))(1+\delta L_{N+1}(T_{N-1}))}$$
$$[\sum_{k=0}^{n-1} w_{N+k}(0)L_{N+k-1}(T_{N-1}) - K]_+\Big|S(T_{N-2})\Big]$$
$$\geq \delta D(T_{N-2},T_N)[\sum_{k=0}^{n-1} w_{N+k}(0)$$
$$E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{(1+\delta L_N(T_{N-1}))(1+\delta L_{N+1}(T_{N-1}))}\Big|S(T_{N-2})\Big] - K]_+. \tag{48}$$

In the same way we evaluate the expectation terms of the equation (48) under the assumption all Brownian motions are uncorrelated each other.

$$E^{T_N}\Big[\frac{L_{N+k-1}(T_{N-1})}{(1+\delta L_N(T_{N-1}))(1+\delta L_{N+1}(T_{N-1}))}\Big|S(T_{N-2})\Big]$$
$$\geq \frac{L_{N+k-1}(T_{N-2})}{(1+\delta L_N(T_{N-2}))(1+\delta L_{N+1}(T_{N-2}))} \tag{49}$$

Hence C has the following relation.

$$C \geq \delta D(T_{N-2}, T_N)[\sum_{k=0}^{n-1} w_{N+k}(0) \frac{L_{N+k-1}(T_{N-2})}{(1+\delta L_N(T_{N-2}))(1+\delta L_{N+1}(T_{N-2}))} - K]_+ \tag{50}$$

Comparing the third terms of the left side on the equation (10) and the right side of equation (50), we have

$$\begin{aligned} [S(T_{N-2}) - K]_+ \leq & \frac{D(T_{N-2}, T_N)}{D(T_{N-2}, T_{N+1})} \\ & [\sum_{k=0}^{n-1} w_{N+k}(0) \frac{L_{N+k-1}(T_{N-2})}{(1+\delta L_N(T_{N-2}))(1+\delta L_{N+1}(T_{N-2}))} - K]_+. \end{aligned} \tag{51}$$

Because $\frac{D(T_{N-2},T_N)}{D(T_{N-2},T_{N+1})} > 1$ one of the sufficient conditions for non early exercise of the Bermudan swaption at $t = T_{N-2}$ derived from the comparison of the third terms is

$$S(T_{N-2}) \leq \prod_{s=3}^{4} \frac{1}{1+\delta L_{N+s-3}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}). \tag{52}$$

In the same way we can derive that the sufficient conditions to satisfy this proposition are summarized as

$$S(T_{N-2}) < \frac{1}{1+\delta L_{N+l-2}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}), l = 1, 2; \tag{53}$$

$$S(T_{N-2}) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{N+s-3}(T_{N-2})} \sum_{k=0}^{n-1} w_{N+k}(0) L_{N+k-1}(T_{N-2}), l = 3, 4, \cdots, n. \tag{54}$$

□

4 Conditions for Non Early Exercise of the Bermudan Swaption at $t = T_i$

In this section we derive theoretical conditions, under which the option holder does not exercise the Bermudan swaption at previous periods, T_i for $i = 0, \cdots, N-1$.

Proposition 4.1. *The holder of the Bermudan swaption does not exercise the option at* $t = T_i$ *for* $i = 0, \cdots, N-1$ *under the conditions*

$$S(T_i) < \frac{1}{1+\delta L_{i+l}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 1, 2; \tag{55}$$

$$S(T_i) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s-1}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 3, 4, \cdots, n. \tag{56}$$

Proof. At $t = T_{N-2}$ from the result of Proposition 1 we prove that we do not exercise the option under the conditions (53) and (54). At $t = T_{i+1}$ we suppose that we do not exercise the option under the conditions

$$S(T_{i+1}) < \frac{1}{1+\delta L_{i+l+1}(T_{i+1})} \sum_{k=0}^{n-1} w_{i+k+3}(0) L_{i+k+2}(T_{i+1}), l = 1, 2; \tag{57}$$

$$S(T_{i+1}) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s}(T_{i+1})} \sum_{k=0}^{n-1} w_{i+k+3}(0) L_{i+k+2}(T_{i+1}), l = 3, 4, \cdots, n, \tag{58}$$

that is

$$\begin{aligned} W(S(T_{i+1})) &= D(T_{i+1}, T_{i+3}) E^{T_{i+3}} \Big[\frac{W(S(T_{i+2}))}{D(T_{i+2}, T_{i+3})} \Big| S(T_{i+1}) \Big] \\ &> \delta [S(T_{i+1}) - K]_+ \sum_{k=0}^{n-1} D(T_{i+1}, T_{i+k+2}). \end{aligned} \tag{59}$$

At $t = T_i$ we would like to show that under the conditions

$$S(T_i) < \frac{1}{1+\delta L_{i+l}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 1, 2; \tag{60}$$

$$S(T_i) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s-1}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 3, 4, \cdots, n, \tag{61}$$

we do not exercise the option by the induction, that is

$$\begin{aligned} & \delta [S(T_i) - K]_+ \sum_{k=0}^{n-1} D(T_i, T_{i+k+1}) \\ &< D(T_i, T_{i+2}) E^{T_{i+2}} \Big[\frac{W(S(T_{i+1}))}{D(T_{i+1}, T_{i+2})} \Big| S(T_i) \Big]. \end{aligned} \tag{62}$$

From the hypothesis, (59), substituting RHS of (59) for the RHS of (62) we obtain

$$\begin{aligned} RHS of (62) &> D(T_i, T_{i+2}) E^{T_{i+2}} \Big[\frac{\delta [S(T_{i+1}) - K]_+ \sum_{m=0}^{n-1} D(T_{i+1}, T_{i+m+2})}{D(T_{i+1}, T_{i+2})} \Big| S(T_i) \Big] \\ &= D(T_i, T_{i+2}) E^{T_{i+2}} \Big[\delta [\sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_{i+1}) - K]_+ \\ & \frac{\sum_{m=0}^{n-1} D(T_{i+1}, T_{i+m+2})}{D(T_{i+1}, T_{i+2})} \Big| S(T_i) \Big]. \end{aligned} \tag{63}$$

Comparing the first terms of RHS of (63) and LHS of (62), we obtain one of the non early exercise conditions as

$$\begin{aligned} & [S(T_i) - K]_+ < \\ & \frac{D(T_i, T_{i+2})}{D(T_i, T_{i+1})} E^{T_{i+2}} \Big[\sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_{i+1}) - K]_+ \Big| S(T_i) \Big]. \end{aligned} \tag{64}$$

Utilizing the following relation

$$\begin{aligned} RHSof(64) &\geq \frac{1}{1+\delta L_{i+1}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) E^{T_{i+2}} \Big[L_{i+k+1}(T_{i+1}) \Big| S(T_i) \Big] - K]_+ \\ &\geq [\frac{1}{1+\delta L_{i+1}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i) - K]_+, \end{aligned} \tag{65}$$

(64) is satisfied under the condition (60). Comparing the second terms of RHS of (63) and LHS of (62), we obtain one of the non early exercise conditions as

$$[S(T_i) - K]_+ < E^{T_{i+2}} \Big[\frac{D(T_{i+1}, T_{i+3})}{D(T_{i+1}, T_{i+2})} [\sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_{i+1}) - K]_+ \Big| S(T_i) \Big]. \tag{66}$$

Utilizing the following relation

$$\begin{aligned} RHSof(66) &\geq [\frac{1}{1+\delta L_{i+2}(T_{i+1})} \sum_{k=0}^{n-1} w_{i+k+2}(0) E^{T_{i+2}} \Big[L_{i+k+1}(T_{i+1}) \Big| S(T_i) \Big] - K]_+ \\ &\geq [\frac{1}{1+\delta L_{i+2}(T_{i+1})} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i) - K]_+, \end{aligned} \tag{67}$$

(66) is satisfied under the condition (60). Comparing the l-th($l \geq 3$) terms of RHS of (63) and LHS of (62), we obtain the one of the non early exercise conditions as

$$\begin{aligned} &[S(T_i) - K]_+ < \\ &\frac{D(T_i, T_{i+2})}{D(T_i, T_{i+l})} E^{T_{i+2}} \Big[\frac{D(T_{i+1}, T_{i+l+1})}{D(T_{i+1}, T_{i+2})} [\sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_{i+1}) - K]_+ \Big| S(T_i) \Big]. \end{aligned} \tag{68}$$

Utilizing the following relation

$$\begin{aligned} RHSof(68) &\geq \prod_{u=0}^{l-3} (1+\delta L_{i+u+2}(T_i)) E^{T_{i+2}} \Big[[\prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s+2}(T_{i+1})} \\ &\quad \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_{i+1}) - K]_+ \\ &\geq [\prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s+2}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i) - K]_+, \end{aligned} \tag{69}$$

(69) is satisfied under the condition (61). Then we prove that the holder of the Bermudan swaption does not exercise the option at $t = T_i$ under the conditions

$$S(T_i) < \frac{1}{1+\delta L_{i+l}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 1, 2; \tag{70}$$

$$S(T_i) < \prod_{s=3}^{l+1} \frac{1}{1+\delta L_{i+s-1}(T_i)} \sum_{k=0}^{n-1} w_{i+k+2}(0) L_{i+k+1}(T_i), l = 3, 4, \cdots, n. \tag{71}$$

□

5 Conclusion

In this paper we propose the conditions for non early exercise of the Bermudan swaption. We derive theoretical conditions, under which the option holder does not exercise the Bermudan swaption. The conditions are derived by optimality equations under varying forward neutral probabilities. We can utilize this property for making profitable exercise strategies.

References

Andersen, L. (1999) A Simple Approach to the Pricing of Bermudan Swaption in the Multifactor LIBOR Market Model, *Journal of Computational Finance*, 3, 2, Winter, 5–32.

Andersen, L. and Andreasen, J. (2001) Factor Dependence of Bermudan Swaptions: Fact or Fiction?, *Journal of Financial Economics*, 62, 3–37.

Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81, 637–654.

Brace, A., Gątarek, D., and Musiela, M. (1997) The Market Model of Interest Rate Dynamics, *Mathematical Finance*, 7, 127–155.

Brigo, D. and Mercurio, F. (2000) *Interest Rate Models Theory and Practice*, Springer–Verlag Berlin Heidelberg New York.

Broadie, M. and Glasserman, P. (1997a) Pricing American-Style Securities Using Simulation, *Journal of Economic Dynamics and Control*, 21, 8–9, 1323–1352.

Broadie, M. and Glasserman, P. (1997b) A Stochastic Mesh Method for Pricing High-Dimensional American Options, Working Paper, Columbia University, New York.

Carr, P. and Yang, G. (1997) Simulating Bermudan Interest Rate Derivatives, Working Paper, Courant Institute at New York University, New York.

Clewlow, L. and Strickland, C. (1998) *Implementing Derivatives Models*, John Wiley and Son Publishing.

Filipović, D. (2002) *Fixed Income Models*, Lecture Note ORF555/FIN555, Department of Operations Research and Financial Engineering, Princeton University, Fall.

Hull, J. and White, A. (1990) Pricing Interest–Rate–Derivative Securities, *Review of Financial Studies*, 3, 4, 573–592.

Hull, J. and White, A. (1994) Numerical Procedures for Implementing Term Structure Models : Single–Factor Models, *The Journal of Derivatives*, Fall, 7–16.

Ito, D., Ohnishi, M. and Tamba, Y. (2004) Pricing of a Chooser Flexible Cap and its Calibration, Working Paper, Osaka University, Osaka.

Jäckel, P. and Rebonato, R. (2003) The Link between Caplet and Swaption Volatilities in a Brace–Gątarek–Musiela/Jamshidian Framework: Approximate Solutions and Empirical Evidence, *Journal of Computational Finance*, Summer, 6, 4, 41–59.

Joshi, M. S. and Rebonato, R. (2003) A Displaced– Diffusion Stochastic Volatility LIBOR Market Model: Motivation, Definition and Implementation, *Quantitative Finance*, 3, 458–469.

Longstaff, F. A. and Schwartz, E. S. (2001) Valuing American Options by Simulation: A Simple Least–Squares Approach, *Review of Financial Studies*, 14, 1, 113–147.

Tamba, Y. (2006) Pricing the Bermudan Swaption with the Efficient Calibration, *NUCB Journal of Economics and Information Science*, 52, 1, 17–31.

Pedersen, M. B. and Sidenius, J. (1998) Valuation of Flexible Caps, *The Journal of Derivatives*, Spring, 60–67.

Rebonato, R. (1998) *Interest Rate Option Models, Second Edition*, Wiley Chichester.

Rebonato, R. (2002) *Modern Pricing of Interest-Rate Derivatives: The Libor Market Model and Beyond*, Princeton University Press.

Rebonato, R. (2004) *Volatility and Correlation: The Perfect Hedger and the Fox*, John Wiley and Sons Publishing.

Rubinstein, M. (1983) Displaced Diffusion Option Pricing, *The Journal of Finance*, March, 38, 1, 213–217.