Learning Effects with a Discrete—Time Approximate Model

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Abstract

In Tamba(2007) I analyzed temporal learning effects of the asset allocation decision of an investor, who has a long investment horizon. The investor has an uncertainty about the mean return of the risky stock (the state variable). In this paper, I discuss the result derived in Tamba(2007) in detail. I verify propriety of a hedge portfolio in the uncertainty. I prove that the partial differential of the investor's expected utility of a bequest in terms of a current assessment of the coefficient is positive in a discrete time approximate model.

keywords: asset allocation; uncertainty; Bayesian learning; dynamic programming
JEL classification: D81; D83; G11

1 Introduction

In Tamba(2007), I considered an asset allocation problem with an uncertainty of a state variable. I analyzed the simple case when the investment opportunity set is constant in time, and the variance of the stock return is known in advance but the expected stock return (the state variable) is uncertain. Brennan (1998) used the same setting and shows the relationship between the investment fraction of the stock and the remaining period by using numerical examples. But I analytically examined the temporal change of the fraction solving the stochastic differential equation of the expected state variables. In this way, I could see the forward looking change of the investment fraction on the stock. I considered the simple model that the investment opportunity set was constant, and the expected return of the stock was unknown in the absence of the estimated risk. By considering the simple case, the temporal change of the investment fraction could be obtained theoretically.

In this paper, I discuss the result derived in Tamba(2007). In Tamba(2007) I derived the result under a plausible conjecture that a partial differential of an expected utility in terms of an expectation of a trend of a stock price is positive. But I did not discuss propriety of the conjecture. In this paper I verify propriety of the conjecture in a discrete–time approximate model.

The paper is organized as follows. In Section 2, I overview the result in the continuous model. In Section 3, I discuss propriety of the conjecture in a discrete–time approximate model. Section 4 concludes the paper.

2 Overview of the Result in Continuous–Time Model

In Tamba(2007), I considered an investor with a long horizon who maximizes the expected bequest at the end of the horizon, T (>0). The investor can trade continuously in a riskless asset or a single risky stock. The real return on the riskfree asset is assumed to be constant, r. The stock price process $(S(t); t \in [0,T])$ is assumed to follow a stochastic differential equation with a drift affected by an unobservable state variable process $(\mu(t); t \in [0,T])$:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\mathrm{d}t + \sigma \mathrm{d}B_1(t), \quad t \in [0,T], \tag{1}$$

where $S(0)$ is a constant and σ $(>0)$is a constant diffusion parameter, and $(\mu(t))$ follows a stochastic differential equation:

$$\mathrm{d}\mu(t) = a\mu(t)\mathrm{d}t + b\mathrm{d}B_2(t), \quad t \in [0,T], \tag{2}$$

where $\mu(0)$ is a random variable, a and b are constant parameters, and $((B_1(t), B_2(t)); t \in [0,T])$ is a two dimensional standard Brownian motion. All of uncertainties in the economy are assumed to be generated by $\mu(0)$ and $((B_1(t), B_2(t))$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $W(t)$ denote the investor's wealth at time $t \in [0,T]$. The stochastic process $(W(t); t \in [0,T])$ is given by:

$$\mathrm{d}W(t) = \alpha(t)W(t)\frac{\mathrm{d}S(t)}{S(t)} + r(1-\alpha(t))W(t)\mathrm{d}t, \quad t \in [0,T].$$

The investor's indirect utility function is characterized on his wealth level $W(t)$, his current assessment of the coefficient $m(t)$, and time t. Therefore, the investor's expected utility of the bequest under an optimal policy is:

$$J(W(t), m(t), t) = \max_{(\alpha(u); t \le u \le T)} \mathbb{E}\left[U(W(T),T) \,\middle|\, \mathcal{F}^S(t)\right], \quad t \in [0,T] \tag{3}$$

with a terminal condition $J(W(T), m(T), T) = U(W(T), T)$.

I assume that the bequest function $U(W(T), T)$ displays Constant Relative Risk Aversion (CRRA):

$$U(W(T),T) = \frac{W(T)^{1-\delta}}{1-\delta}, \tag{4}$$

where $\delta > 0$ is the degree of relative risk aversion. Finally, I derived the following proposition in Tamba(2007).

Proposition 1. *The optimal investment fraction on the stock, $\alpha(t)$ at time $t \in [0,T]$ can be represented as:*

$$\alpha(t) = \frac{m(t) - r + \gamma(t)\dfrac{\Phi_m(m(t),t)}{\Phi(m(t),t)}}{\delta\sigma^2}, \quad t \in [0,T]. \tag{5}$$

□

Under the conjecture that

$$J_m(W(t),m(t),t) > 0, \quad t \in [0,T] \tag{6}$$

and the case of $\delta > 1$, I have

$$\Phi_m(m(t),t) < 0, \quad t \in [0,T],$$

because

$$J_m(W(t),m(t),t) = \frac{W(t)^{1-\delta}}{1-\delta}\Phi_m(m(t),t) > 0, \quad t \in [0,T].$$

Further, I have

$$\Phi(m(t),t) > 0, \quad t \in [0,T]$$

because

$$J(W(t),m(t),t) = \frac{W(t)^{1-\delta}}{1-\delta}\Phi(m(t),t) < 0, \quad t \in [0,T]. \tag{7}$$

$J_m(W(t),m(t),t) > 0$, $t \in [0,T]$ is a plausible conjecture. Hence, under this conjecture, I have

$$\frac{\Phi_m(m(t),t)}{\Phi(m(t),t)} < 0, \quad t \in [0,T].$$

I can consider that $\gamma(t)\Phi_m(m(t),t)/\Phi(m(t),t)$ is the hedge for the uncertainty of μ.

3 A Discrete–Time Approximate Model

In this paper, I verify whether the conjecture,

$$J_m(W(t),m(t),t) > 0, \quad t \in [0,T]$$

is appropriate. In order to do this, I consider an approximate model in a discrete–time setting. When I can observe the stock price process, $(S(t); 0 \leq$

$t \leq T)$ with a time interval $\Delta > 0$. Since $(S(t); 0 \leq t \leq T)$ follows the geometric Brownian motion,

$$X_n^\Delta := \ln \frac{S((n+1)\Delta)}{S(n\Delta)} = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left\{B_1((n+1)\Delta) - B_1(n\Delta)\right\}, \quad n = 1, 2, \cdots, \frac{T}{\Delta}, \tag{8}$$

where T/Δ is assumed to be an integer, for convenience. Hence,

$$X_n^\Delta \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta, \sigma^2\Delta\right), \quad n = 1, 2, \cdots, \frac{T}{\Delta}. \tag{9}$$

Let's consider a simple case first. Assume that a random variable X has a normal distribution $N(C, v)$ and that the mean C is also a random variable which has a normal prior distribution $N(M_n, \Gamma_n)$.

Hence, the probability density functions of X and C are given by

$$\mathbb{P}(X \in \mathrm{d}x | C = c) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(x-c)^2}{2v}\right)\mathrm{d}x, \quad x \in \mathbb{R}; \tag{10}$$

$$\mathbb{P}(C \in \mathrm{d}c) = \frac{1}{\sqrt{2\pi\Gamma_n}} \exp\left(-\frac{(c-M_n)^2}{2\Gamma_n}\right)\mathrm{d}c, \quad c \in \mathbb{R}, \tag{11}$$

respectively. So, by the Bayes formula, the posterior probability density function of C given an observation $\{X = x\}$ $(x \in \mathbb{R})$ becomes:

$$\begin{aligned}\mathbb{P}(C \in \mathrm{d}c | X = x)\mathrm{d}x &\propto \mathbb{P}(C \in \mathrm{d}c)\mathbb{P}(X \in \mathrm{d}x | C = c) \\ &\propto \exp\left(-\frac{1}{2}\left\{\frac{(c-M_n)^2}{\Gamma_n} + \frac{(x-c)^2}{v}\right\}\right)\mathrm{d}c \cdot \mathrm{d}x, \\ & c \in \mathbb{R}.\end{aligned} \tag{12}$$

Since

$$\begin{aligned}&\frac{(c-M_n)^2}{\Gamma_n} + \frac{(x-c)^2}{v} \\ =\ & \frac{1}{\Gamma_n v}\left\{v(c-M_n)^2 + \Gamma_n(c-x)^2\right\} \\ =\ & \frac{1}{\Gamma_n v}\left\{(v+\Gamma_n)c^2 - 2(vM_n + \Gamma_n x)c + vM_n^2 + \Gamma_n x^2\right\} \\ =\ & \frac{1}{\Gamma_n v}\{(v+\Gamma_n)\left(c - \frac{vM_n + \Gamma_n x}{v+\Gamma_n}\right)^2 \\ & -\frac{(vM_n + \Gamma_n x)^2}{v+\Gamma_n} + vM_n^2 + \Gamma_n x^2\},\end{aligned} \tag{13}$$

I have

$$\mathbb{P}(C \in \mathrm{d}c | X = x) \propto \exp\left(-\frac{1}{2\frac{\Gamma_n v}{v+\Gamma_n}}\left(c - \frac{vM_n + \Gamma_n x}{v+\Gamma_n}\right)^2\right)\mathrm{d}c, \quad c \in \mathbb{R}.$$

$$(14)$$

In order to apply the above result to our model in which the stock price follows a (discrete–time) geometric Brownian motion, I set

$$C = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta; \tag{15}$$

$$v = \sigma^2\Delta. \tag{16}$$

Let me define the conditional expectation and variance by

$$\begin{aligned} M_n &:= \mathbb{E}\left[C \,\middle|\mathcal{F}_n^S\right] \\ &= \mathbb{E}\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta\middle|\,\mathcal{F}_n^S\right] \\ &= \left(m_n - \frac{1}{2}\sigma^2\right)\Delta, \quad n = 0, 1, \cdots, \frac{T}{\Delta}; \end{aligned} \tag{17}$$

$$\begin{aligned} \Gamma_n &:= \mathbb{E}\left[(C - M_n)^2|\mathcal{F}_n^S\right] \\ &= \mathbb{E}\left[\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta - \left(m_n - \frac{1}{2}\sigma^2\right)\Delta\right\}^2\middle|\,\mathcal{F}_n^S\right] \\ &= \mathbb{E}\left[(\mu - m_n)^2|\mathcal{F}_n^S\right]\Delta^2 \\ &= \gamma_n\Delta^2, \quad n = 0, 1, \cdots, \frac{T}{\Delta}, \end{aligned} \tag{18}$$

where, for $n = 0, 1, \cdots, T/\Delta$,

$$\begin{aligned} \mathcal{F}_n^S &:= \sigma\left(S(0), S(\Delta), \cdots, S(n\Delta)\right); \\ m_n &:= \mathbb{E}[\mu|\mathcal{F}_n^S]; \\ \gamma_n &:= \mathbb{E}[(\mu - m_n)^2|\mathcal{F}_n^S]. \end{aligned}$$

Then, according to the previous results, they can be updated by the following formulas:

$$M_{n+1} = \frac{vM_n + \Gamma_n x}{v + \Gamma_n}, \quad n = 0, 1, \cdots, \frac{T}{\Delta} - 1; \tag{19}$$

$$\Gamma_{n+1} = \frac{\Gamma_n v}{v + \Gamma_n}, \quad n = 0, 1, \cdots, \frac{T}{\Delta} - 1. \tag{20}$$

From Eqs. (17) and (18), Eq. (19) becomes

$$\left(m_{n+1} - \frac{1}{2}\sigma^2\right)\Delta = \frac{\sigma^2\Delta\left(m_n - \frac{1}{2}\sigma^2\right)\Delta + \gamma_n\Delta^2 X_n^\Delta}{\sigma^2\Delta + \gamma_n\Delta^2}. \tag{21}$$

Substituting (8) to (21), I have

$$\left(m_{n+1}-\frac{1}{2}\sigma^2\right)\Delta$$
$$=\frac{\sigma^2\Delta\left(m_n-\frac{1}{2}\sigma^2\right)\Delta+\gamma_n\Delta^2\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\Delta+\sigma(B_1((n+1)\Delta)-B_1(n\Delta))\right\}}{\sigma^2\Delta+\gamma_n\Delta^2} \quad (22)$$

As a result, I can get the following stochastic difference equation:

$$m_{n+1}-m_n=\frac{\gamma_n}{\sigma^2+\gamma_n\Delta}\left\{(\mu-m_n)\Delta+\sigma\left\{B_1((n+1)\Delta)-B_1(n\Delta)\right\}\right\}. \quad (23)$$

Substituting Eqs. (16) and (18) to Eq. (20), I have

$$\gamma_{n+1}=\frac{\gamma_n\sigma^2}{\sigma^2+\gamma_n\Delta}. \quad (24)$$

Hence I get the following (deterministic) difference:

$$\gamma_{n+1}-\gamma_n=-\frac{\gamma_n^2\Delta}{\sigma^2+\gamma_n\Delta}. \quad (25)$$

From (24), (25), I can conclude that this discrete–time model closely approximates the continuous–time model when Δ is sufficiently small.

The investor optimizes his expected bequest at the end of the time horizon, T. Let $V(W_n,m_n,n)$ be the optimal value function for this discrete–time model when the process starts from the state (W_n,m_n,n). Then, the optimality equation is:

$$V(W_n,m_n,n)=\max_{\alpha\in A}\mathbb{E}\left[V(W_{n+1},m_{n+1},n+1)\,\middle|\mathcal{F}_n^S\right],\quad n=0,1,\cdots,\frac{T}{\Delta}-1, \quad (26)$$

with a terminal condition $V(W_N,m_N,N)=U(W_N,T)$ for $N:=T/\Delta$, where $A\subset\mathbb{R}$ is a constraint set on stock position.

Proposition 2. *m_{n+1} increases in m_n and X_n^Δ.*

Proof. From (21), this is clearly true. □

Proposition 3. *If $A\subset\mathbb{R}_+:=[0,\infty)$, W_{n+1} increases in X_n^Δ.*

Proof. Note that

$$\begin{aligned} W_{n+1} &= f(W_n,\alpha_n,X_n^\Delta) \\ &:= \alpha_nW_n\exp\left(X_n^\Delta\right)+(1-\alpha_n)W_n(1+r) \\ &= \alpha_nW_n\frac{S((n+1)\Delta)}{S(n\Delta)}+(1-\alpha_n)W_n(1+r) \end{aligned} \quad (27)$$

From the assumption, $\alpha_n\geq 0$, I can conclude that W_{n+1} increases in X_n^Δ. □

Proposition 4. *$V(W_n, m_n, n)$ increases in W_n.*

Proof. Suppose that $W_n < \widehat{W}_n$. The investor who has $\widehat{W}_n$ can spend W_n to attain $V(W_n, m_n, n)$ and invest the remaining $\widehat{W}_n - W_n$ for the riskless bond by which the investor can obtain the additional utility for sure. Hence $V(W_n, m_n, n) < V(\widehat{W}_n, m_n, n)$. □

Proposition 5. *If $A \subset \mathbb{R}_+$ $\mathbb{P}$–a.s., then $V(W_n, m_n, n)$ increases in m_n.*

Proof. The optimality equation can be rewritten as

$$\begin{aligned} & V(W_n, m_n, n) \\ = \; & \max_{\alpha \in A} \mathbb{E}\left[V(W_{n+1}, m_{n+1}, n+1) \,\middle|\, \mathcal{F}_n^S\right] \\ = \; & \max_{\alpha \in A} \int V\left(f(W_n, \alpha, x), h(m_n, \gamma_n, x, n+1), n+1\right) p(x|m_n, \gamma_n, n) \mathrm{d}x, \end{aligned} \tag{28}$$

where

$$\begin{aligned} f(W_n, \alpha, x) &:= \alpha W_n \mathrm{e}^x + (1-\alpha) W_n (1+r); \\ h(m_n, \gamma_n, x, n+1) &:= \frac{1}{\sigma^2 + \gamma_n \Delta}\left[\sigma^2 m_n + \gamma_n x + \frac{1}{2}\gamma_n \sigma^2 \Delta\right]; \\ p(x|m_n, \gamma_n, n) &:= \frac{1}{\sqrt{2\pi\gamma_n}} \exp\left(-\frac{(x-m_n)^2}{2\gamma_n}\right). \end{aligned} \tag{29}$$

It is noted that, when γ_n is fixed and $m_n < \widehat{m}_n$, p.d.f. $p(\cdot|\widehat{m}_n, \gamma_n, n)$ is greater than p.d.f. $p(\cdot|m_n, \gamma_n, n)$ in the sense of the first order stochastic dominance (usual stochastic order). From Proposition 4, $W_{n+1} = f(W_n, \alpha, x)$ increases in x. From the Propositions 3, 4, and 5, it suffices to show that $V(W_{n+1}, m_{n+1}, n+1)$ increases in m_{n+1}.

By induction in the case of $n' = N := T/\Delta$,

$$V(W_N, m_N, N) = U(W_N, T) \tag{30}$$

is increasing in W_N.

When $n' = n+1$, suppose that $V(W_{n+1}, m_{n+1}, n+1)$ increases in m_{n+1}.

Then, when $n' = n$,

$$V(W_n, m_n, n) = \max_{\alpha \in A} \int V\left(f(W_n, \alpha, x), h(m_n, \gamma_n, x, n+1), n+1\right) p(x|m_n, \gamma_n, n) \mathrm{d}x. \tag{31}$$

Because $m_{n+1} = h(m_n, \gamma_n, x, n+1)$ increases in m_n, $V(W_n, m_n, n)$ increases in m_n. Since this discrete-time model closely approximates the continuous-time model, I conclude that the conjecture, $J_m(W(t), m(t), t) > 0, \quad t \in [0, T]$, is appropriate. □

4 Conclusion

In Tamba(2007), I analyzed the case that the investment opportunity set was unknown but constant. Especially, I considered the case that the volatility of the stock return was known, but the expected return of the stock was unknown. I theoretically analyzed the temporal change of the fraction of investment on the risky asset. I concluded that he learning process gave two effects on the investment fraction. One was improving the assessment of the state variable. The other was the reduction of the hedge demand against the uncertainty of the state variable learning about the state variable. I derived the result under a plausible conjecture that a partial differential of an expected utility in terms of an expectation of a trend of a stock price was positive.

In this paper, I discuss the result derived in Tamba(2007). I verify propriety of the conjecture in a discrete–time approximate model. I confirm propriety of the important conjecture in the discussion of Tamba(2007).

References

[1] Brennan, M. J.(1998) "The Role of Learning in Dynamic Portfolio Decisions," *European Financial Review*, **1**, 296–306.

[2] Brennan, M. J., Schwartz, E. S., and Lagnado, R. (1997) "Strategic Asset Allocation," *Journal of Economic Dynamics and Control*, **21**, 1377–1403.

[3] Detemple, J. B. (1986) "Asset Pricing in an Economy with Incomplete Information," *Journal of Finance*, **41**, 383–392.

[4] Dothan, U., and Feldman D. (1986) "Equilibrium Interest Rates and Multiperiod Bonds in a Partially Observable Economy," *Journal of Finance*, **41**, 369–382.

[5] Duffie, D. (2002) *Dynamic Asset Pricing Theory*, 3rd Ed., Princeton, Princeton University Press.

[6] Gennotte, G. (1986) "Optimal Portfolio Choice under Incomplete Information," *Journal of Finance*, **41**, 733–749.

[7] Ibbotson Associates, Inc., *1995 Yearbook*.

[8] Kandel, S., and Stambaugh, R. F. (1996) "On the Predictability of Stock Returns: An Asset–Allocation Perspective," *Journal of Finance*, **51**, 385–424.

[9] Liptser, R. S., and Shiryayev, A. N. (2000) *Statistics of Random Processes*, Vols. 1, 2, Springer–Verlag, New York.

[10] Merton, R. C. (1971) "Optimum Consumption and Portfolio Rules in a Continuous–Time Model," *Journal of Economic Theory*, **3**, 373–413.

[11] Tamba, Y. (2007) "Dynamic Asset Allocation under Uncertainty with Continuous Time Model," *NUCB Journalof Economics and Information Science*, **52**, 1, 45–55.

[12] Xia, Y. (2001) " Learning about Predictability: The Effect of Parameter Uncertainty on Asset Allocation," *Journal of Finance*, **56**, 205–246.