A study of periodic review two-level supply chain inventory model

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Abstract

We consider a single product, two-stage supply chain inventory model with the upstream warehouse and the downstream retailer which is observed periodically. The demand for a product is random. The optimal supply chain inventory policy to minimize the total discounted expected cost is derived via dynamic programming. The problem is analyzed in single, two, multiple and infinite periods. Under certain conditions, we show that the problem for the retailer is a Newsboy-type problem and that the optimal policy is characterized by a single critical number for the initial supply chain inventory level. We further show that in all cases, the optimal policy for the warehouse is a base-stock policy where the optimal base-stock level depends on the initial supply chain inventory level. Numerical analysis is examined to gain insight into the problem.
Keywords : inventory, dynamic programming, periodic review models, supply chain management

1 Introduction

The term supply chain refers to a system consisting of material suppliers, production facilities, distribution services, and customers who are all linked together via the downstream feedback flow of materials (deliveries) and the upstream feedback flow of orders (Stevens 1989; Disney and Towill 2003). In many supply chains, the variance of orders may be considerably larger than the variance of sales, and this distortion tends to increase as one moves upstream, this is so-called “Bullwhip Effect” The Bullwhip phenomenon has recognized in many diverse markets. Procter & Gamble found that the diaper orders issued by the distributors have a degree of variability that cannot be explained by consumer demand fluctuations (Lee, Padamanabhan and Wang 1997a). This phenomenon was extensively analyzed by Lee, Padamanabhan and Wang (1997a, b), which have pointed out five fundamental causes, demand signal processing (information sharing), order batching, rational game, price variations and long lead time. Thus, the members in a supply chain have been facing such a phenomenon that causes the increasing of the average inventory and the total expected cost. In order to avoid the bullwhip effect, we

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should eliminate or decrease those causes.

In traditional supply chain, the downstream makes stock level decision and the upstream only observes the downstream's order. But, with the development of information technology (IT) such as Internet, point-of-sales (POS), and electronic data interchange (EDI), all stages of the supply chain have been able to share demand and inventory data quickly and inexpensively. So, recently, there have been some new approaches that transfer the stock level decision to the upstream, such as Click and Mortar (CAM), Drop-shipping (DS), and Vendor managed inventory (VMI).

There are numerous works published in the area of SCM. We will only discuss literatures that relate closely to our paper. There is an extensive literature on serial inventory system. Clark and Scarf[8] consider a multi-echelon inventory problem. They show that echelon base-stock policies are optimal in a periodic review, finite-horizon setting. Federgruen and Zipkin[9] generalyze the same results to periodic review, infinite-horizon models. Chen and Zheng[6] consider supply chain with random demand, constant lead time, and setup cost at all stages. They derive a lower bound over all feasible policies under centralized control. Recently, Chen[5] develop a serial inventory system with N stages. With centralized demand information, he shows that the optimal echelon reorder points can be determined sequentially. Axsäter[1] derives complete probability distributions for the retailer inventory level in a two-echelon distribution inventory system. Çetinkaya and Lee[4] present a model for coordination of inventory and transportation decisions in VMI systems. They approximate the exact model to obtain a solution to the considered problem. It is necessary to understand concept of environment dependent optimal inventory policy in periodic review model. It will be find in the appendix in this paper.

In this paper, we introduce a two-stage, supply chain inventory model with the upstream warehouse and the downstream retailer. At the beginning of a period, a retailer provides demand information and inventory level to a warehouse. Then, the warehouse decides the replenishment quantity for retailer and the order quantity to minimize the total supply chain discounted expected cost. We can see such situation in the real-life, e.g., the soft drink firms that replenish their product for the vending machines, many electronics/computer industry, and the garment industry, etc.

The purpose of this paper is to show that under certain conditions, the optimal policy for the retailer is characterized by a single critical number for the initial supply chain inventory level at each period, which is obtained by solving a myopic cost function. We further show that the optimal policy for the warehouse is a base-stock policy where the optimal base-stock level depends on the initial supply chain inventory level at each period. In this paper, in particular, we focus on the finite-horizon analysis, since it gives us concrete and realistic insight.

This paper is organized as follows; Section 2 presents the formulation of the general problem as a dynamic programming model. Section 3 and 4 provide analyses for the single-period and the multi-period problems, respectively. Numerical examples are presented in Section 5. The paper concludes with some final remarks in Section 6.

2 Assumption and Notation

This paper studies a two-stage supply chain inventory model with the upstream warehouse and the downstream retailer as illustrated in Figure 1.

In this section, we introduce notations and basic assumptions used throughout the paper:

n : number of periods remaining in the finite-horizon problem;

D_n : the random variable which describes the total demand during period n;

$A(z)=P\{D_n \le z\}$: the cumulative distribution function of D_n;

$a(z)$: the probability density function corresponding to $A(z)$;

λ : a mean of the demand, i.e., $\lambda=\int_0^\infty zdA(z)=\int_0^\infty za(z)dz$;

W : warehouse;

R : retailer;

x_W^n : the inventory level at W observed at the beginning of period n;

x_R^n : the inventory level at R observed at the beginning of period n;

x_T^n : the total supply chain inventory level observed at the beginning of period n, i.e., $x_T^n=x_W^n+x_R^n$;

y_W^n : the order-up-to level at W for period n;

y_R^n : the order-up-to level at R for period n;

y_T^n : the total supply chain order-up-to level for period n, i.e., $y_T^n=y_W^n+y_R^n$;

c_W : a unit ordering cost at W;

h_W : a unit holding cost at W incurred at the end of period;

c_D : a unit transportation cost from W to R;

c_D^0 : a unit transportation cost from W to R in period 0;

h_R : a unit holding cost at R incurred at the end of period;

p_R : a unit shortage cost at R incurred at the end of period;

α : the discount factor per period, $0< \alpha <1$;

We assume that $A(0)=0$, $a(\cdot)> 0$, that unsatisfied demands are fully backlogged, and that the transportation lead-time is negligible. Also, we make the following assumptions.

Assumption 1 $c_W + h_W + \alpha c_D < \alpha p_R$

Assumption 2 $c_D + h_R + h_W < \alpha c_D$

Assumption 1 is necessary to motivate ordering. If Assumption 2 does not hold, it is less expensive to transport all products at W to R. Then, the problem becomes trivial.

Since we do not allow for disposing of any inventory without satisfying demand, the admissibility condition requires as follows:

Admissibility condition 1 $y_R^n - x_R^n \leq x_W^n$, i.e., $x_W^n \leq x_T^n$

Admissibility condition 2 $y_R^n \leq x_R^n$

Admissibility condition 3 $y_W^n \leq x_W^n - (y_R^n - x_R^n)$, i.e., $y_T^n \leq x_T^n$

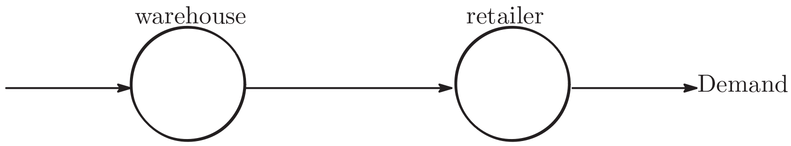

Figure 1: A two-stage supply chain

Now, let $V^n(x_R, x_T)$ be the minimum total supply chain discount expected cost of operating for n-period with the initial inventory level x_R at R, and the initial supply chain inventory level x_T under the best ordering/replenishment decision is used at period n through period 1. Then, a dynamic programming equation (DPE) for the problem can be given by

$$V^0(x_R, x_T) = 0, \tag{1}$$

$$V^n(x_R, x_T) = \min_{y_T \geq x_T \geq y_R \geq x_R} \left\{ c_D(y_R - x_R) + y_T(h_W + c_W) - h_W y_R - c_W x_T + L(y_R) + \alpha \int_0^\infty V^{n-1}(y_R - z, y_T - z) dA(z) \right\}, \; n > 0, \tag{2}$$

where y_R and y_T are the inventory level after the replenishment at R and the supply chain inventory level after the order is delivered, respectively,

$$L(y_R) = h_R \int_0^{y_R} (y_R - z) dA(z) + p_R \int_{y_R}^\infty (z - y_R) dA(z), \;\; n > 0$$

is the expected one-period holding and shortage cost function at R. The first and second derivatives of $L(y_R)$ are

$$L'(y_R) = (h_R + p_R)A(y_R) - p_R,$$
$$L''(y_R) = (h_R + p_R)a(y_R) > 0.$$

To simplify our analysis, by using the relation

$$W^n(x_R, x_T) = V^n(x_R, x_T) + c_D x_R, \;\; n \geq 0,$$

we change (1) and (2) to following DPE.

$$\begin{aligned} W^0(x_R, x_T) &= c_D^0 x_R, \\ W^n(x_R, x_T) &= \min_{y_T \geq x_T \geq y_R \geq x_R} \left\{ f(y_R) + y_T(h_W + c_W) \right. \\ &+ \left. \alpha \int_0^\infty W^{n-1}(y_R - z, y_T - z) dA(z) \right\} \\ &+ \alpha c_D \lambda - c_W x_T \end{aligned} \tag{3}$$

where

$$f(y_R) = c_D y_R (1 - \alpha) + L(y_R) - h_W y_R \tag{4}$$

In this paper, we assume that no action is taken in period 0. So, c_D^0=0. Thus,

$$W^0(x_R, x_T) = 0.$$

We assume that all parameters and costs are nonnegative, and that all relevant functions are differentiable.

3 Single-Period Analysis

In this section, we analyze the single-period problem for the model introduced in the last section. Furthermore, we make an additional assumption, and change (3) to a DPE which is a function of the initial supply chain inventory level x_T only. We begin by rewriting (3) as

$$W^1(x_R, x_T) = \min_{y_T \geq x_T} \{y_T(h_W + c_W)\} + F(x_R, x_T) + \alpha c_D \lambda - c_W x_T \quad (5)$$

where

$$F(x_R, x_T) = \min_{x_T \geq y_R \geq x_R} \{f(y_R)\} \quad (6)$$

We first investigate the properties of (4) since it plays a central role in the minimization in (5) and (6). We obtain the first two derivatives as follows:

$$\begin{aligned} f'(y_R) &= c_D + L'(y_R) - h_W, \\ f''(y_R) &= L''(y_R) > 0. \end{aligned}$$

It should be noted that $L'(\cdot)$ is increasing,

$$\begin{aligned} \lim_{y_R \to \infty} f'(y_R) &= c_D(1-\alpha) + h_R - h_W > 0, \\ \lim_{y \to -\infty} f'(y_R) &= c_D(1-\alpha) - p_R - h_W < 0. \end{aligned}$$

So, there exists a unique y_R^f such that $f'(y_R)$=0, i.e.,

$$y_R^f = A^{-1}\left[\frac{p_R + h_W - c_D(1-\alpha)}{h_R + p_R}\right].$$

y_R^f is nonnegative and finite because $p_R+h_W - c_D(1-\alpha)>0$ and $p_R+h_W - c_D(1-\alpha)<h_R+p_R$.

Now, we set an additional assumption.

Assumption 3 $x_R \leq y_R^f$

Then, the optimal policy for period 1 and the expected optimal cost under the optimal policy are summarized in the following theorem.

Theorem 1 *For 1-period problem,*
(1) *the optimal replenishment policy for R is given by*

$$y_R^* = \begin{cases} x_T & (x_T < y_R^f) \\ y_R^f & (x_T \geq y_R^f) \end{cases}$$

where critical number y_R^f is a solution to $f'(y_R)$=0;
(2) *obviously, it is optimal not to order at W, i.e.,*

$$y_T^{1*} = x_T;$$

(3) *the optimal cost under this policy is*

$$W^1(x_T) = F(x_T) + \alpha c_D \lambda + h_W x_T$$

where

$$F(x_T) = \begin{cases} f(x_T) & (x_T < y_R^f) \\ f(y_R^f) & (x_T \geq y_R^f) \end{cases}$$

and its first two derivatives are

$$\begin{aligned} W^{1\prime}(x_T) &= F'(x_T) + h_W \\ &= \begin{cases} f'(x_T) + h_W (x_T < y_R^f) \\ h_W (x_T \geq y_R^f) \end{cases} \\ W^{1\prime\prime}(x_T) &= F''(x_T) \\ &= \begin{cases} f''(x_T)(x_T < y_R^f) \\ 0(x_T \geq y_R^f) \end{cases} \geq 0. \end{aligned}$$

So, $W^1(x_T)$ and $F(x_T)$ are quasi-convex in x_T.

Under Assumption 3,notice that (5)and (6)are func-tions of the initial supply chain inventory level x_T only. Hence, recursively,we can rewrite (3)as

$$\begin{aligned} W^n(x_T) &= \min_{y_T \geq x_T} \left\{ G^n(y_T) \right\} + F(x_T) \\ &+ \alpha c_D \lambda - c_W x_T \end{aligned} \tag{7}$$

where

$$\begin{aligned} G^n(y_T) &= y_T(h_W + c_W) \\ &+ \alpha \int_0^\infty W^{n-1}(y_T - z) dA(z) \end{aligned} \tag{8}$$

Remark 1 *It should be noted that y_R has no effect on either y_T or the cost-to-go. So, the optimal replenishment policy for R is a myopic solution which depends only on the initial supply chain inventory level x_T as defined in Theorem 1.*

Remark 2 *It is not unreal that the initial inventory level x_R at R is less than the maximum inventory level y_R^f at R. Furthermore, the replenishment policy is easy to implement, since it is obtained by solving a myopic cost function.*

In the next section, we show that the optimal ordering policy at *W* is a base-stock policy where the optimal base-stock level depends on the initial supply chain inventory level.

4 Multi-Period Analysis

In this section, we analyze the two-, *n*-, and infinite-period problem for the model introduced in Section 2.

When *n*=2, from (7) and (8),

$$\begin{aligned} W^2(x_T) &= \min_{y_T \geq x_T} \left\{ G^2(y_T) \right\} + F(x_T) + \alpha c_D \lambda - c_W x_T \\ G^2(y_T) &= y_T(h_W + c_W) + \alpha \int_0^\infty W^1(y_T - z) dA(z) \end{aligned}$$

The key results for the two-period problem are contained in

Theorem 2 *The optimal ordering policy at W for the two-period model is a base-stock policy defined by*

$$y_T^{2*} = \begin{cases} S_T^2 & (x_T \leq S_T^2) \\ x_T & (x_T > S_T^2) \end{cases} \tag{9}$$

where S_T^2 *satisfies* $(h_W + c_W) + \alpha \int_0^\infty W^{1\prime}(y_T - z)dA(z) = 0$ *and the optimal cost incurred by this policy is*

$$W^2(x_T) = \begin{cases} G^2(S_T^2) + F(x_T) + \alpha c_D \lambda - c_W x_T \ (x_T \leq S_T^2) \\ \\ G^2(x_T) + F(x_T) + \alpha c_D \lambda - c_W x_T \ (x_T > S_T^2) \end{cases} \tag{10}$$

Moreover, $W^2(x_T)$ is quasi-convex in x_T.

Proof. Let us analyze the objective function $G^2(y_T)$. We obtain the first two derivatives of $G^2(y_T)$ as follows:

$$\begin{aligned} G^{2\prime}(y_T) &= (h_W + c_W) + \alpha \int_0^\infty W^{1\prime}(y_T - z)dA(z) \\ &= (1+\alpha)h_W + c_W + \alpha \int_{y_T - y_R^f}^\infty f'(y_T - z)dA(z) \\ G^{2\prime\prime}(y_T) &= \alpha \int_0^\infty W^{1\prime\prime}(y_T - z)dA(z) \\ &= \alpha \int_{y_T - y_R^f}^\infty (h_R + p_R)a(y_T - z)dA(z) \end{aligned}$$

Then,

$$\begin{aligned} & G^{2\prime\prime}(y_T) > 0, \\ \lim_{y_T \to \infty} & G^{2\prime}(y_T) &=& \ (1+\alpha)h_W + c_W > 0 \\ \lim_{y_T \to -\infty} & G^{2\prime}(y_T) &=& \ h_W + c_W + \alpha c_D(1-\alpha) \\ & &-& \ p_R < 0 \end{aligned}$$

So, there exists a unique S_T^2 such that $G^{2\prime}(y_T)=0$. It is clear from above results that S_T^2 is the optimal order-up-to level when $x_T \leq S_T^2$ and no order should be placed when $x_T > S_T^2$. This proves that a base-stock policy defined by y_T^{2*} in (9) is optimal. Accordingly, the optimal total cost $W^2(x_T)$ is found by evaluating $G^2(y_T)$ at y_T^{2*} so that

$$W^2(x_T) = \begin{cases} G^2(S_T^2) + F(x_T) + \alpha c_D \lambda - c_W x_T \ (x_T \leq S_T^2) \\ \\ G^2(x_T) + F(x_T) + \alpha c_D \lambda - c_W x_T \ (x_T > S_T^2) \end{cases}$$

which leads to (10). Furthermore,

$$W^{2\prime}(x_T) = \begin{cases} F'(x_T) - c_W & (x_T \le S_T^2) \\ G^{2\prime}(x_T) + F'(x_T) - c_W & (x_T > S_T^2) \end{cases}$$

$$W^{2\prime\prime}(x_T) = \begin{cases} F''(x_T) & (x_T \le S_T^2) \\ G^{2\prime\prime}(x_T) + F''(x_T) & (x_T > S_T^2) \end{cases} \ge 0$$

So, $W^2(x_T)$ is quasi-convex in x_T. Q.E.D.

For the n-period problem, the DPE is given by (7) and (8). The key results for the n-period problem are contained in

Theorem 3 *The optimal ordering policy at W for the n-period model is a base-stock policy defined by*

$$y_T^{n*} = \begin{cases} S_T^n & (x_T \le S_T^n) \\ x_T & (x_T > S_T^n) \end{cases} \tag{11}$$

where S_T^n *satisfies* $(h_W + c_W) + \alpha \int_0^\infty W^{n-1\prime}(y_T - z)dA(z) = 0$ *and the optimal cost incurred by this policy is*

$$W^n(x_T) = \begin{cases} G^n(S_T^n) + F(x_T)\lambda + \alpha c_D - c_W x_T \, (x_T \le S_T^n) \\ G^n(x_T) + F(x_T)\lambda + \alpha c_D - c_W x_T \, (x_T > S_T^n) \end{cases} \tag{12}$$

Moreover, $W^n(x_T)$ *is quasi-convex in* x_T*.*

Proof. For the n-period problem, the objective function is given by (8). Aassume that the following properties hold for the $(n-1)$-period problem:

(1) $G^{n-1}(y_T)$ is convex and attains its global minimum at S_T^{n-1}.

(2) $\lim_{y_T \to \infty} G^{n-1\prime}(y_T) = c_W + h_W \sum_{k=0}^{n-2} \alpha^k$

(3) $W^{n-1}(x_T)$ is given in (12) for $n=n-1$ and is quasi-convex in x_T.

Then, it can be shown inductively that similar properties as in the two-period case also hold for the n-period problem. Q.E.D.

We develop and solve the infinite-horizon problem. For the infinite-horizon problem, the DPE which is equivalent to (7) can be written as

$$W(x_T) = \min_{y_T \ge x_T} \left\{ G(y_T) \right\} + F(x_T) + \alpha c_D \lambda - c_W x_T \tag{13}$$

where

$$G(y_T) = y_T(h_W + c_W) + \alpha \int_0^\infty W(y_T - z)dA(z) \tag{14}$$

According to Proposition 14 in Bertsekas[2], under the positivity assumption (i.e., expected costs per period are nonnegative), we have $\lim_{n\to\infty} W^n(x_T) = W(x_T)$ and $\lim_{n\to\infty} G^n(y_T) = G(y_T)$. Moreover, there exists a stationary optimal policy. Thus, the optimal cost in infinite-horizon is characterized by (13) and there exists a stationary policy y_T^* which minimizes the infinite-horizon total cost. The key results for the infinite-horizon problem are given in

Theorem 4 *The optimal ordering policy at W for the infinite-horizon model is a stationary base-stock policy defined by*

$$y_T^* = \begin{cases} S_T & (x_T \le S_T) \\ x_T & (x_T > S_T) \end{cases} \tag{15}$$

where S_T satisfies $(h_W + c_W) + \alpha \int_0^\infty W'(y_T - z)dA(z) = 0$ and the optimal cost incurred by this policy is

$$W(x_T) = \begin{cases} G(S_T) + F(x_T) + \alpha c_D \lambda - c_W x_T \; (x_T \le S_T) \\ G(x_T) + F(x_T) + \alpha c_D \lambda - c_W x_T \; (x_T > S_T) \end{cases} \tag{16}$$

Moreover, $W(x_T)$ is quasi-convex in x_T.

Proof. For the infinite-horizon problem, the objective function is given by (14). Lemma 8-4 and 8-5 of Heyman and Sobel[10] imply $G'(y_T) = \lim_{n\to\infty} G'(y_T)$. Then, by using similar discussions as in the proof for the Theorem 2 and 3, we obtain the desired results. Q.E.D.

5 Numerical Illustrations

In this section, we compute y_R^f and S_T^n which characterize the optimal policy for our model. We note that, in our numerical examples, y_R^f and S_T^n are rounded to the nearest integer.

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
91	187	219	231	233	233	233	233	233	233

Table 1: y_R^f and S_T^n for the base parameter values

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$\lambda = 20$					
36	75	88	92	93	93	93	93	93	93
				$\lambda = 100$					
182	375	439	462	466	466	466	466	466	466

Table 2: y_R^f and S_T^n for varying λ values

We assume the following base values for the parameters in the model:
n=10, λ =50 (with exponential demand), c_W=5, h_W=5, c_D=15, h_R=10, p_R=30, α =0.9.
The results are displayed in Table1.

To observe the effect of the parameter values on y_R^f and S_T^n, we provide additional numerical examples by varying each parameter with the others kept at their original base values. The results are reported in Table2-8 and summarized in the following:

- As the demand increases stochastically, y_R^f as well as S_T^n increase.
- An increase in c_W leads to a decrease in S_T^n.
- An increase in h_W leads to an increase in y_R^f and a decrease in S_T^n.
- An increase in c_D leads to a decrease in y_R.
- An increase in h_R leads to a decrease in y_R^f and an increase in S_T^n.
- An increase in p_R leads to an increase in y_R^f as well as in S_T^n.
- An increase in α leads to an increase in y_R^f as well as in S_T^n.

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$c_W = 2$					
91	198	226	235	236	236	236	236	236	236
				$c_W = 8$					
91	178	213	227	230	231	231	231	231	231

Table 3: y_R^f and S_T^n for varying c_W values

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$h_W = 2$					
72	210	248	264	269	269	269	269	269	269
				$h_W = 8$					
122	173	203	213	214	214	214	214	214	214

Table 4: y_R^f and S_T^n for varying h_W values

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$c_D = 12$					
93	187	220	231	233	233	233	233	233	233
				$c_D = 18$					
89	187	219	231	233	233	233	233	233	233

Table 5: y_R^f and S_T^n for varying c_D values

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$h_R = 7$					
118	185	217	229	231	231	231	231	231	231
				$h_R = 13$					
75	190	222	233	235	236	236	236	236	236

Table 6: y_R^f and S_T^n for varying h_R values

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$p_R = 27$					
87	183	215	227	229	229	229	229	229	229
				$p_R = 33$					
94	191	223	234	237	237	237	237	237	237

Table 7: y_R^f and S_T^n for varying p_R values

y_R^f	S_T^2	S_T^3	S_T^4	S_T^5	S_T^6	S_T^7	S_T^8	S_T^9	S_T^{10}
				$\alpha = 0.8$					
80	183	213	222	223	223	223	223	223	223
				$\alpha = 0.99$					
102	191	225	238	242	242	242	242	242	242

Table 8: y_R^f and S_T^n for varying α values

6 Concluding Remarks

In this paper, we consider a single product, two-stage supply chain inventory model with the upstream warehouse W and the downstrean retailer R which is observed periodically. Under certain conditions, we show that the optimal replenishment policy for R is a myopic solution which depends only on the initial supply chain inventory level. Furthermore, we clarify that the optimal ordering policy at W is a base-stock policy where the optimal base-stock level depends on the initial supply chain inventory level. Numerical examples are provided to gain insight into the problem. The derived policy is easy to implement. The derived policy is easy to implement.

Our results are quite satisfactory and well-defined.

An extension that we are currently working on is the case where it is necessary to coordinate inventory and transportation. If there are a fixed cost, positive lead-time, and finite capacity to replenish, it may be economical to hold small replenishment until a consolidated quantity accumulates, even though it must be paid shortage costs. That is, it is an important issue to balance the trade-off between scale economies associated with transportation and customer waiting.

Finally, We are considering the application of meta-heuristic methods like neural network and genetic algorithms as a implementation to this problem.

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Appendix

A Environment dependent optimal inventory policy

A.1 Assumption and Notation

Consider a single-product periodic-review inventory system for N-periods. Let the period be numbered such that the final period is denoted as period 1, while the first period is denoted as period N.

The state of the environment observed at the beginning of period n (n=1,2,l... ,N) is represented by I_n and we assume that I={I_n;$n \geq 0$ } is Markov chain on a countable state space E with a given transition matrix P=$P(i,j)$=$P[I_{n+1}=j|I_n=i]$. Let X_n denote the inventory level observed at the beginning of period n. The basic assumption of this model is that the demand distribution and the cost-parameters at any period depend on the state of the environment at the beginning of that period. Therefore, the decision maker observes both the inventory level and the environment state to decide on the optimal order quantity which is delivered immediately.

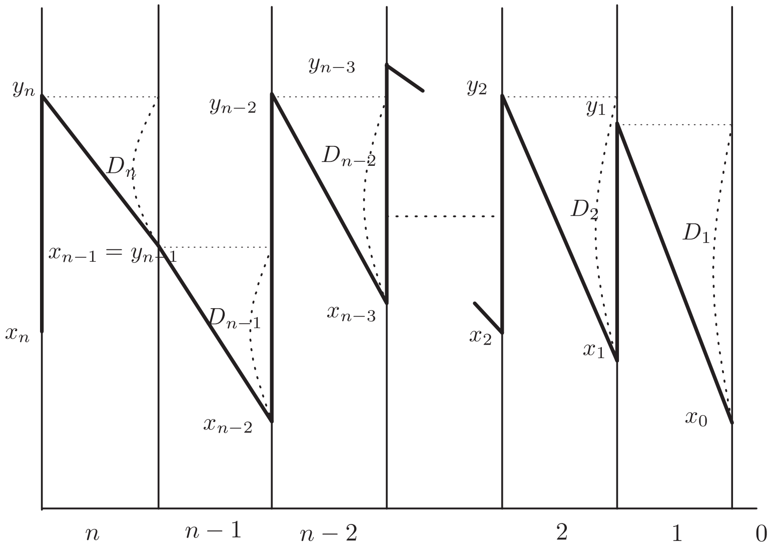

Figure 2: The behavior of the inventory level

If D_n is the total demand during period n, then the demand process D={D$_n$; $n \geq 0$ } is depend on the Markov chain I so that its conditional distribution function is $A_i(z_n)$=P[$D_n \leq z_n|I_n=i$], with the probability density function $a_i(z_n)$. Also, we assume $A_i(0)$=0, ai(cdot)>0.

We consider the following four types of costs: if the environmental state is i, a fixed ordering cost K_i independent of the order quantity, a unit ordering cost c_i, a unit holding cost h_i incurred at the end of period, and a unit shortage cost p_i incurred at the end of period. To motivate ordering, we assume that p_i>c_i as in standard models. Also, we assume that unsatisfied demands are fully backlogged.

Let $Y_n(i, x_n)$ be the order-up-to level if the environment is i and the inventory level is x_n at the beginning of period n. The admissibility condition requires that $Y_n(i, x_n) \geq x_n$ since we do not allow for discarding of any inventory without satisfying demand. It is noted that, for any y_n, the inventory level X_n is a Markov chain, where

$$X_{n-1}=x_n+[y_n(i,x_n)-x_n]^+-D_n$$

for $n \geq 0$. Figure 1 illustrates the behavior of the inventory level.

Now, let $V_i^n(x_n)$ be the minimum expected total discount cost of operating for n-period with the state of the environment i and the initial inventory level x_n, under the best ordering decision is used at period n through period 1. Then, a dynamic programming equation (DPE) for the problem can be given as

$$V_i^0(x_0) \equiv 0,$$

and

$$V_i^n(x_n) = \min_{y_n \geq x_n} \{K_i \delta(y_n - x_n) + G_i^n(y_n) - c_i x_n\},\ n > 0, i \in E, \tag{17}$$

where

$$\delta(y_n - x_n) = \begin{cases} 1 & \text{if } y_n - x_n > 0, \\ 0 & \text{if } y_n - x_n = 0, \end{cases}$$

and

$$\begin{aligned} G_i^n(y_n) &= c_i y_n + L_i^n(y_n) \\ &+ \alpha \sum_{j \in E} P(i,j) \int_0^\infty V_j^{n-1}(y_n - z_n) dA_i(z_n), n > 0, i \in E \end{aligned} \tag{18}$$

with the expected holding and shortage cost function at period n

$$L_i^n(y_n) = h_i \int_0^{y_n} (y_n - z_n) dA_i(z_n) + p_i \int_{y_n}^\infty (z_n - y_n) dA_i(z_n)$$

and the discount factor α per period.

The decision variable in this model is y_n, so (18) plays a central role to find the optimal value y_n^*.

We assume that all parameters and costs are nonnegative, and that all relevant functions are differentiable.

A.2 single-period analysis

This analysis will provide important insights in understanding the two-period analysis and n-period analysis. We begin by rewriting (17) and (18) as

$$V_i^1(x_1) = \min_{y_1 \geq x_1} \{K_i \delta(y_1 - x_1) + G_i^1(y_1) - c_i x_1\}, \tag{19}$$

$$G_i^1(y_1) = c_i y_1 + L_i^1(y_1), \tag{20}$$

where

$$L_i^1(y_1) = h_i \int_0^{y_1} (y_1 - z_1) dA_i(z_1) + p_i \int_{y_1}^\infty (z_1 - y_1) dA_i(z_1)$$

We first investigate the properties of (20) since it plays a central role in the minimization in (19). We obtain the first two derivatives of (20) as follows:

$$\frac{dG_i^1(y_1)}{dy_1} = G_i'^1(y_1) = c_i + (h_i + p_i)A_i(y_1) - p_i, \quad (21)$$

$$\frac{d^2G_i^1(y_1)}{dy_1^2} = G_i''^1(y_1) = (h_i + p_i)a_i(y_1). \quad (22)$$

Then,

$$G_i''^1(y_1) = (h_i + p_i)a_i(y_1) > 0, \quad (23)$$

because h_i, p_i, and $a_i(y_1) > 0$. So, $G_i^1(y_1)$ is convex in y_1. It should be noted that the rightside of (21) is increasing in y_1,

$$\lim_{y_1 \to \infty} G_i'^1(y_1) = c_i + h_i > 0 \quad (24)$$

and

$$\lim_{y_1 \to 0} G_i'^1(y_1) = c_i - p_i < 0. \quad (25)$$

Therefore, there exists a unique solution such that

$$G_i'^1(y_1) = c_i + (h_i + p_i)A_i(y_1) - p_i = 0. \quad (26)$$

Let S_i^1 solve (26), that is,

$$S_i^1 = A_i^{-1}\left[\frac{p_i - c_i}{h_i + p_i}\right].$$

S_i^1 is nonnegative and finite because $0 < \left[\frac{p_i - c_i}{h_i + p_i}\right] < 1$ with $(p_i - c_i) > 0$ and $(p_i - c_i) < (h_i + p_i)$.

Now the property of (20) can be characterized by using (21) and (22):

(1) For $y_1 < S_i^1$, $G_i'^1(y_1) > 0$, $G_i''^1(y_1) > 0$, hence $G_i^1(y_1)$ is decreasing and convex.

(2) For $y_1 > S_i^1$, $G_i'^1(y_1) > 0$, $G_i''^1(y_1) > 0$, hence $G_i^1(y_1)$ is increasing and convex.

¿From these observations, it is clear that $G_i^1(y_1)$ attains its global minimum at $y_1 = S_i^1$ with value $G_i^1(S_i^1)$.

Now, consider the minimization in (20), in particular the term $(K_i\,\delta\,(y_1 - x_1) + G_i^1(y_1)$. The nature of $(K_i + G_i^1(y_1))$ is identical to that of $G_i^1(y_1)$ and it attains the global minimum at S_i^1 with value $K_i + G_i^1(S_i^1)$.

For $y_1 \le S_i^1$, since $G_i^1(y_1)$ is decreasing in y_1, there exists a unique solution such that

$$K_i + G_i^1(S_i^1) = G_i^1(y_1) \quad (27)$$

Let S_i^1 solve (27), then it follows from the definition of S_i^1 and the decreasing property of $G_i^1(y_1)$ for $y_1 \le S_i^1$ that

$$K_i + G_i^1(S_i^1) \le G_i^1(y_1)\ for\ y_1 \le s_i^1, \quad (28)$$

and

$$K_i + G_i^1(S_i^1) > G_i^1(y_1)\ for\ s_i^1 < y_1 \le S_i^1. \quad (29)$$

The behavior of $G_i^1(y_1)$ and $G_i^1(y_1) + K_i$ is shown graphically in Figure 3.

Based upon the state of the environment i and the initial inventory level x_1, the optimal policy can now be characterized in terms of the two critical numbers S_i^1 and s_i^1. For $x_1 \leq s_i^1$, the advantage $(G_i^1(x_1) - G_i^1(S_i^1))$ gained by ordering up to S_i^1 can offset the fixed ordering cost K_i provided one plans to order. This follows from (28). On the other hand, for $s_i^1 < x_1 \leq S_i^1$, it is not worthwhile to order because the fixed ordering cost K_i will offset the expected savings $(G_i^1(x_1) - G_i^1(S_i^1))$ derived from ordering $(S_i^1 - x_1)$ units. This follows from (29). Since ordering will increase the expected cost for $x_1 > S_i^1$, it is not worthwhile to order, too.

Now the following summarizes the optimal policy for period 1 and the property of $G_i^1(y_1)$.

(1) the optimal policy for period 1 is given by

$$Y_1^*(i, x_1) = \begin{cases} S_i^1 & \text{if } x_1 \leq s_i^1, \\ x_1 & \text{if } x_1 > s_i^1, \end{cases}$$

where critical numbers S_i^1 and s_i^1 are solutions to Eqs.(26) and (27) and are the order-up-to point and the reorder point, respectively.

(2) $G_i^1(y_1)$ is convex and $\begin{cases} \text{decreasing in } y_1 & \text{if } y_1 \leq S_i^1, \\ \text{increasing in } y_1 & \text{if } y_1 > S_i^1. \end{cases}$

Therefore, the expected cost $V_i^1(x_1)$ under the optimal policy is obtained by substituting $Y_1^*(i, x_1)$ into (3):

$$V_i^1(x_1) = \begin{cases} K_i + G_i^1(S_i^1) - c_i x_1 & \text{if } x_1 \leq s_i^1, \\ G_i^1(x_1) - c_i x_1 & \text{if } x_1 > s_i^1. \end{cases} \tag{30}$$

And its first two derivatives are

$$V_i'^1(x_1) = \begin{cases} -c_i & \text{if } x_1 \leq s_i^1, \\ G_i'^1(x_1) - c_i & \text{if } x_1 > s_i^1, \end{cases} \tag{31}$$

$$V_i''^1(x_1) = \begin{cases} 0 & \text{if } x_1 \leq s_i^1, \\ G_i''^1(x_1) & \text{if } x_1 > s_i^1. \end{cases} \tag{32}$$

So, $V_i^1(x_1)$ is quasi-convex in x_1.

A.3 n-period analysis

To use induction, we assume that the following properties hold for the (n-1)-period problem, where the state of the environment is $j \in E$.

$$Y_{n-1}^*(j, x_{n-1}) = \begin{cases} S_j^{n-1} \\ \text{if } x_{n-1} \leq s_j^{n-1}, \\ x_{n-1} \\ \text{if } x_{n-1} > s_j^{n-1}, \end{cases} \tag{33}$$

$$G_j'^{n-1}(S_j^{n-1}) = 0 \tag{34}$$

$$K_i + G_j^{n-1}(S_j^{n-1}) = G_j^{n-1}(s_j^{n-1}) \tag{35}$$

$$G_j''^{n-1}(\cdot) > 0 \tag{36}$$

$$\lim_{y_{n-1}\to\infty} G_j'^{n-1}(y_{n-1}) = c_j + h_j + \alpha \sum_{k\in E} P(j,k)h_k$$
$$+\alpha^2 \sum_{k,l\in E} P(j,k)P(k,l)h_l + \cdots + \alpha^{n-2} \sum P(j,k)P(k,l)$$
$$\cdots P(\chi,\psi)P(\psi,\omega)h_\omega > 0 \tag{37}$$

$$\lim_{y_{n-1}\to 0} G_j'^{n-1}(y_{n-1}) < c_j$$
$$-\alpha \sum_{k\in E} P(j,k)c_k - p_j < 0 \tag{38}$$

$$V_j^{n-1}(x_{n-1}) = \begin{cases} K_j + G_j^{n-1}(S_j^{n-1}) - c_j x_{n-1} \\ \text{if } x_{n-1} \le s_j^{n-1}, \\ G_j^{n-1}(x_{n-1}) - c_j x_{n-1} \\ \text{if } x_{n-1} > s_j^{n-1}. \end{cases} \tag{39}$$

$$V_j'^{n-1}(x_{n-1}) = \begin{cases} -c_j \\ \text{if } x_{n-1} \le s_j^{n-1}, \\ G_j'^{n-1}(x_{n-1}) - c_j \\ \text{if } x_{n-1} > s_j^{n-1}. \end{cases} \tag{40}$$

$$V_j''^{n-1}(x_{n-1}) = \begin{cases} 0 \\ \text{if } x_{n-1} \le s_j^{n-1}, \\ G_j''^{n-1}(x_{n-1}) \\ \text{if } x_{n-1} > s_j^{n-1}. \end{cases} \tag{41}$$

For an n-period problem, the DPE is given by (1). We investigate the property of (2) since it plays a central role in the minimization in (1).

First, we rewrite (2) by substituting V_j^{n-1} from (39) into it as

$$G_i^n(y_n) = y_n\{c_i - \alpha \sum_{j\in E} P(i,j)c_j\} + L_i^n(y_n)$$
$$+\alpha \sum_{j\in E} P(i,j)\Big[\int_{y_n - s_j^{n-1}}^{\infty} \Big\{K_j + G_j^{n-1}(S_j^{n-1})$$
$$-c_j z_n\Big\} dA_i(z_n) + \int_0^{y_n - s_j^{n-1}} \Big\{G_j^{n-1}(y_n - z_n)$$
$$-c_j z_n\Big\} dA_i(z_n)\Big]. \tag{42}$$

And, from (40),(41), we obtain the first and the second order derivatives for (42) as follows:

$$G_i'^n(y_n) = c_i - \alpha \sum_{j\in E} P(i,j)c_j$$
$$+(h_i + p_i)A_i(y_n) - p_i + \alpha \sum_{j\in E} P(i,j)$$
$$\times \int_0^{y_n - s_j^{n-1}} G_j'^{n-1}(y_n - z_n) dA_i(z_n). \tag{43}$$

$$G_i''^n(y_n) = (h_i + p_i)a_i(y_n)$$
$$+\alpha \sum_{j \in E} P(i,j) \int_0^{y_n - s_j^{n-1}} G_j''^{n-1}(y_n - z_n) dA_i(z_n). \tag{44}$$

Then,

$$G_i''^n(y_n) = (h_i + p_i)a_i(y_n)$$
$$+\alpha \sum_{j \in E} P(i,j) \int_0^{y_n - s_j^{n-1}} G_j''^{n-1}(y_n - z_n) dA_i(z_n) > 0. \tag{45}$$

because h_i,p_i,$a_i(y_n)$, and $G_j''^{n-1}(\cdot) > 0$. So, $G_i^n(y_n)$ is convex in y_n. The rightside of (43) is increasing in y_n,

$$\lim_{y_n \to \infty} G_i'^n(y_n) = c_i + h_i + \alpha \sum_{j \in E} P(i,j)h_j$$
$$+\alpha^2 \sum_{j,k \in E} P(i,j)P(j,k)h_k$$
$$+\alpha^3 \sum_{j,k,l \in E} P(i,j)P(j,k)P(k,l)h_l$$
$$+ \cdots + \alpha^{n-1} \sum P(i,j)$$
$$\times P(j,k)P(k,l) \cdots P(\chi,\psi)P(\psi,\omega)h_\omega > 0, \tag{46}$$

and

$$\lim_{y_n \to 0} G_i'^n(y_n) = c_i - \alpha \sum_{j \in E} P(i,j)c_j - p_i$$
$$+\alpha \sum_{j \in E} P(i,j) \int_0^{-s_j^{n-1}} G_j'^{n-1}(-z_n)$$
$$dA_i(z_n) < c_i - \alpha \sum_{j \in E} P(i,j)c_j - p_i < 0. \tag{47}$$

Therefore, there exists a unique solution such that

$$G_i'^n(y_n) = c_i - \alpha \sum_{j \in E} P(i,j)c_j$$
$$+(h_i + p_i)A_i(y_n) - p_i$$
$$+\alpha \sum_{j \in E} P(i,j) \int_0^{y_n - s_j^{n-1}} G_j'^{n-1}(y_n - z_n) dA_i(z_n) = 0. \tag{48}$$

Let S_i^n solve (48) Then, the property of (2) is as follows:

(1) For $y_n \le S_i^n$, $G_j'^n(y_n) \le 0$, $G_j''^n(y_n) > 0$, hence $G_i^n(y_n)$ is decreasing and convex.

(2) For $y_n > S_i^n$, $G_j'^n(y_n) > 0$, $G_j''^n(y_n) > 0$, hence $G_i^n(y_n)$ is increasing and convex.

So, $G_i^1(y_1)$ attains its global minimum at $y_n = S_i^n$ with value $G_i^n(S_i^n)$.

Now, consider the minimization in (1), in particular the term $(K_i\, \delta\, (y_n - x_n) + G_i^n(y_n))$. The nature of $(K_i + G_i^n(y_n))$ is identical to that of $G_i^n(y_n)$ and it attains the global minimum at S_i^n with value $K_i + G_i^n(S_i^n)$. For $y_n \le S_i^n$, since $G_i^n(y_n)$ is decreasing in y_n, there exists a unique solution such that

$$K_i + G_i^n(S_i^n) = G_i^n(y_n) \tag{49}$$

Let s_i^n solve (49), then it follows from the definition of s_i^n and the decreasing property of $G_i^n(y_n)$ for $y_n \leq S_i^n$ that

$$K_i + G_i^n(S_i^n) \leq G_i^n(y_n) \text{ for } y_n \leq s_i^n, \tag{50}$$
$$K_i + G_i^n(S_i^n) > G_i^n(y_n) \text{ for } s_i^n < y_n \leq S_i^n. \tag{51}$$

The behavior of $G_i^n(y_n)$ and $G_i^n(y_n) + K_i$ is shown graphically in Figure 4.

So, based upon the state of the environment i and the initial inventory level x_n, the optimal policy can now be characterized in terms of the two critical numbers S_i^n and s_i^n. For $x_n \leq s_i^n$, the advantage $(G_i^n(x_n) - G_i^n(S_i^n))$ gained by ordering up to S_i^n can offset the fixed ordering cost K_i provided one plans to order. This follows from (50). On the other hand, for $s_i^n < x_n \leq S_i^n$, it is not worthwhile to order because the fixed ordering cost K_i will offset the expected savings $(G_i^n(x_n) - G_i^n(S_i^n))$ derived from ordering $(S_i^n - x_n)$ units. This follows from (51). Since the ordering will increase the expected cost for $x_n > S_i^n$, it is not worthwhile to order, too.

Now the following summarizes the optimal policy for period n and the property of $G_i^n(y_n)$.
(1) the optimal policy for period n is

$$Y_n^*(i, x_n) = \begin{cases} S_i^n & \text{if } x_n \leq s_i^n, \\ x_n & \text{if } x_n > s_i^n, \end{cases}$$

where critical numbers S_i^n and s_i^n are solutions to Eqs.(48) and (49) and are the order-up-to point and the reorder point, respectively.

(2) $S_i^n(y_n)$ is convex and $\begin{cases} \text{decreasing in } y_n & \text{if } y_n \leq S_i^n, \\ \text{increasing in } y_n & \text{if } y_n > S_i^n. \end{cases}$

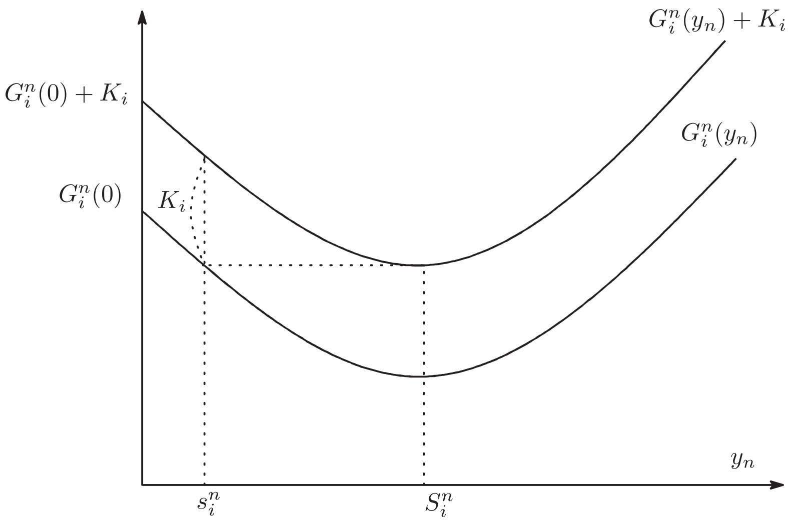

Figure 4: The form of $G_i^n(y_n)$ and $K_i + G_i^n(y_n)$

Therefore, the expected cost $V_i^n(x_n)$ under the optimal policy is obtained by substituting $Y_n^*(i,x_n)$ into (1):

$$V_i^n(x_n) = \begin{cases} K_i + G_i^n(S_i^n) - c_i x_n & \text{if } x_n \le s_i^n, \\ G_i^n(x_n) - c_i x_n & \text{if } x_n > s_i^n. \end{cases} \tag{52}$$

And its first two derivatives are given by

$$V_i'^n(x_n) = \begin{cases} -c_i & \text{if } x_n \le s_i^n, \\ G_i'^n(x_n) - c_i & \text{if } x_n > s_i^n, \end{cases} \tag{53}$$

$$V_i''^n(x_n) = \begin{cases} 0 & \text{if } x_n \le s_i^n, \\ G_i''^n(x_n) & \text{if } x_n > s_i^n. \end{cases} \tag{54}$$

So, $V_i^n(x_n)$ is quasi-convex in x_n.