An analysis of the bid-ask spread for risk-averse traders in a limit-order market*

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Abstract

In this paper, I analyze the relationship between the equilibrium spread and the trader' s risk aversion coefficient, which represents the degree of risk aversion in a limit-order market. I extend Handa et al. (2003) to examine the effect of risk aversion on the equilibrium spread in the market. I show that an increase in risk aversion widens the spread. Handa et al. (2003) show empirically that the spread depends on the proportion of buyers in the market. This empirical result is consistent with the theoretical result derived in this paper.
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1 Introduction

Markets are divided into two types of systems. One is the dealer market, where investors buy at the dealer's ask price and sell at the dealer's bid price. The other is the limit order market, where investors buy (sell) at the limit sell (buy) price, which has been previously placed. The former system is in place in the NASDAQ Stock Market and the London Stock Exchange. The latter system is evidenced by the New York Stock Exchange (NYSE), the Paris Bourse, and the Tokyo Stock Exchange.

It is commonly noted that the limit order market system has less liquidity, but less spread, than the dealer market system. As the dealer, who is the liquidity supplier in the dealer market, is more responsible for providing immediate execution than the limit order trader as liquidity supplier in the limit order market, the supplier in the former requires larger payments. Though academic research has discussed which system is better, there is no definitive answer. For example, in an empirical study comparing the limit order system with the dealer market system, Huang and Stoll (1995) conclude that the execution cost in NASDAQ (a dealer system), is twice as large as that in the NYSE (a limit order system). It follows that the dealer market system causes wider spreads than the limit order market system in order to manage the market. In a theoretical study of dealer markets, Glosten and Milgrom (1985) and Easley and O'Hara (1987) analyze the bid price, ask price, and their spread.

In their model, it is assumed that the dealer sets the present price (bid and ask) given the ex ante

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trade, either sell or buy. While Foucault (1999) presents the framework of a limit order system, Handa et al. (2003) extend this by incorporating the risk-neutral information trader to observe the effect of asymmetric information. In this model, the uninformed risk-neutral trader quotes bid and ask prices under the condition that he or she is indifferent between the revenue obtained via the market order and the expected revenue from the limit order. This paper analyzes the spread in the limit order market by assuming that the uninformed trader has risk-averse utility to see the effect of risk on the spread.

The paper is organized as follows. In the next section, I discuss the limit order market model. I derive the equilibrium price of the bid and ask, and their spread in section 3. In section 4, I numerically examine the relationship between the degree of risk aversion and the equilibrium bid-ask spread. The conclusions are presented in section 5.

2 Limit order market model

Subrahmanyam (1991) analyzes the relationship between market liquidity and the degree of risk aversion for the dealer (liquidity supplier) with risk-averse utility in the dealer market, and concludes that the increment in the risk-averse parameter makes the market less liquid. This paper examines the relationship between the spread and the risk-averse parameter in the limit order market. Therefore, I assume that the uninformed trader, who is the liquidity supplier in the limit order market, has risk-averse utility. For analysis of the limit order market, I note the difference between the limit order market and the dealer market; i.e., the trader in the limit order market trades by either market order or limit order, while the trader in the dealer market trades by market order only.

The difference between the market order and limit order in the limit market is as follows. The trader can choose a market buy (sell) order if the sell (buy) side in the limit order book is not empty, and his or her order is instantaneously executed at the quote price on the sell (buy) side. If he or she submits a limit buy (sell) order, it is not necessarily executed. The limit buy (sell) order is executed at the specified price that he or she posted, only if someone submits the market sell (buy) order, while the limit buy (sell) order expires if all traders post a limit order. Thus, the execution probability of the trader depends on the order type of the other traders. In the following subsections, I indicate the market mechanism and the strategies of informed and uninformed traders.

2.1 Market mechanism

In the market, there are four types of trader, while the market participants are divided into two groups: a buyer group and a seller group. The buyer (seller) group attaches high (low) value V_h (V_l) to the risky asset. Thus, V_h (V_l) is interpreted as the reservation price for the buyer (seller) group. Additionally, the traders in the buyer (seller) group are divided into two trader types: informed and uninformed traders. The proportions of the buyer group and the seller group in the market are defined as k and $1-k$, respectively. Similarly, I define the proportion of the informed trader in each group as δ. The trader trades one risky asset in the following manner.

The buyer (seller) arrives in the market and trades one risky asset by either a market order or a limit order at time 0. If the buyer (seller) decides to submit a market order, then the order is executed at time

0. Thus, the wealth that the buyer (seller) gains on the trading is given by:

$$\widetilde{W} = \begin{cases} V_h - A^M & \text{if the buyer submits market order,} \\ B^M - V_l & \text{if the seller submits market order,} \end{cases}$$

where A^M and B^M are the ask price and bid price at time 0, respectivery. If the buyer (seller) submits a limit order, then the order is not executed at time 0, but would be executed at time 1. The wealth that the buyer (seller) gains at time 1 on trading is determined by the next trader who arrives to trade the risky asset by either market order or limit order. If the next trader submits a market sell (buy) order, then the wealth of the buyer (seller) who submits the limit order at time 0, is given at time 1 by:

$$\widetilde{W} = \begin{cases} V_h + \tilde{\epsilon} - P_B^L & \text{if the buyer submits limit order at time 0,} \\ P_S^L - V_l - \tilde{\epsilon} & \text{if the seller submits limit order at time 0,} \end{cases}$$

where P_B^L (P_S^L) is the specified price at which the limit buy (sell) order is submitted at time 0. The random variable $\tilde{\epsilon}$ represents the private information, which takes a value of $+H$ or $-H$ with equal probability. The only informed trader can observe the private information ex ante. For simplicity, it is assumed that if the next trader submits a limit order, the limit order which the trader submits at time 0 expires; i.e., if the limit order holds for only one period then the wealth for the trader who submits the limit order is 0. It is assumed that all parameters (k, δ, V_h, V_l and H) are known to all traders at time 0.

2.2 Strategy of the informed trader

Because the informed buyer realizes the value of the risky asset as $V_h + H$ ex ante, he or she submits a market buy order if and only if $V_h + H > A^M$ where A^M is the ask price. Similarly, the informed seller who realizes the value of risky asset as $V_l - H$ ex ante submits a market sell order if and only if $V_l - H < B^M$ where B^M is the bid price. If he or she submits the limit order, his or her private information would be revealed because he or she sets the specified price at which the limit order is submitted, giving the private information. Thus, the informed trader submits the market order only to protect private information.

The informed trader is also assumed to be risk neutral; his or her utility is $V_h + H - A^M$ when he or she submits the market buy order. Similarly, it is $B^M - V_l + H$ for the market sell order. It is also assumed that the magnitude of the private signal $\tilde{\epsilon}$ satisfies $V_h - H < A^M$ and $V_l + H > B^M$, and that the informed seller (buyer) switches to the buyer (seller) with probability γ. The probability γ is 1 when the informed seller (buyer), identifies the private signal that satisfies $A^M < V_l + H$ ($V_h - H < B^M$), switches to buyer (seller).

2.3 Strategy of the uninformed trader

The uninformed buyer (seller) expects the value of the risky asset as $V_h (V_l)$; i.e., $E[V_h + \tilde{\epsilon}] = V_h$ ($E[V_l + \epsilon] = V_l$) since he or she does not know whether the private information $\tilde{\epsilon}$ takes $+H$ or $-H$ exante. If the uninformed buyer submits a market order, the order is instantaneously executed at the ask price A^M independently of the private information. The uninformed buyer's wealth is then given by:

$$\widetilde{W} = V_h - A^M.$$

However, the ask price may be undesirable. The limit order is used in order to execute the order at a desirable price. The limit buy order, which the uninformed trader submits at time 0, could be executed at time 1 in cases where the incoming trader is (1) an uninformed seller who submits a market order, (2) an informed seller who observes the value of the risky asset as $V_l - H < P_B^L$ ex ante, and (3) an informed buyer who switches to a seller by observing the value as $V_h - H < P_B^L$ ex ante. The uninformed buyer's wealth is then given by:

$$\widetilde{W} = \begin{cases} V_h + \widetilde{\epsilon} - P_B^L & \text{if case (1)} \\ V_h - H - P_B^L & \text{if case (2) or case (3),} \end{cases}$$

where P_B^L is the specified price at which the limit buy order is submitted at time 0. However, the limit buy order expires at time 1 in cases where the incoming trader is (4) an uninformed seller who submits a limit order, (5) an uninformed buyer, (6) an informed buyer who does not switch to a seller. For simplicity, if the limit order expires, wealth is normalized to zero. The uninformed buyer's wealth is thus given by:

$$\widetilde{W} = 0 \qquad \text{any of three cases.}$$

As described above, the order is executed if the incoming trader is an informed trader who observes $V_l - H < P_B^L$ or is an uninformed seller who submits a market order. The uninformed buyer who submits a limit order cannot control which type of trader will arrive next; i.e., whether the incoming trader is a seller or a buyer and informed or uninformed. However, the limit order which he or she submits would make the incoming uninformed seller's wealth indifferent between a market order and a limit order; i.e., the bid price is determined to make the incoming uninformed seller submit a market sell order. Similarly, the uninformed seller faces a situation similar to the uninformed buyer. If he or she submits a market sell order, then his or her wealth is given by:

$$\widetilde{W} = B^M - V_l,$$

and if he or she submits a limit sell order at specified price P_S^L, he or she gains the following wealth:

$$\widetilde{W} = \begin{cases} 0 & \text{the limit order expires,} \\ P_S^L - V_l - H & \text{the limit order executes } (\widetilde{\epsilon} = +H), \\ P_S^L - V_l + H & \text{the limit order executes } (\widetilde{\epsilon} = -H). \end{cases}$$

The decision tree for the uninformed trader is shown in Figure 1.

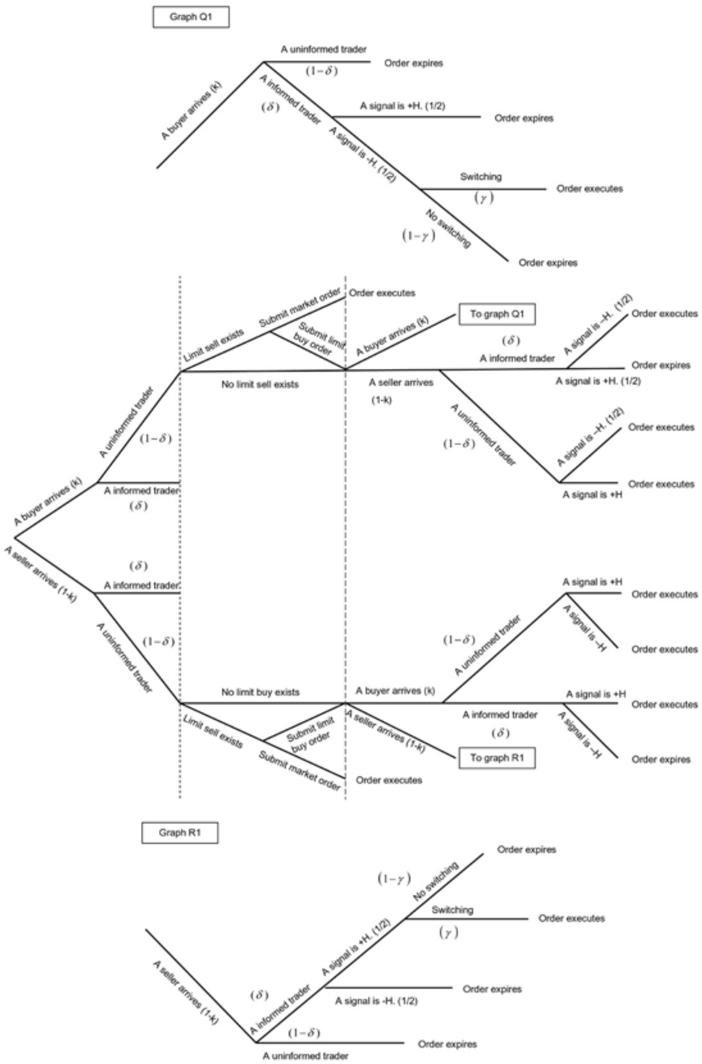

Figure 1: Decision tree for uninformed traders.

In order to introduce the risk of trading into the uninformed trader's utility, theuninformed trader is also assumed to have negative exponential utility with risk-aversion coefficient R, which is known to all traders:

$$U(\widetilde{W}) = -\exp(-R\widetilde{W})$$

If a buyer submits a market order, the utility for the buyer is $U(V_h - A^M)$ where A^M is the ask price. If a buyer submits the limit order at a specified price P_B^L, the expected utility for the buyer is $E[U] = \phi_B\, U(V_h + \widetilde{\epsilon} - P_B^L)$ where $\widetilde{\epsilon}$ is the private information that takes $+H$ or $-H$ with equal probability and ϕ_B is the probability that the limit buy order is executed. The execution probability is given by:

$$\phi_B = \begin{cases} \frac{1}{2}\delta(1-k) + k\ (1-\frac{1}{2}\delta\gamma) & \text{if } U\ (0), \\ \frac{1}{2}(1-\delta)\ (1-k) & \text{if } U\ \ (V_h + H - P_B^L) \\ \frac{1}{2}(1-K) + \frac{1}{2}\delta\gamma & \text{if } U\ (V_h + H - P_B^L) \end{cases}$$

Similarly, the utility of the seller who submits a market order is $U(B^M - V_l)$ where B^M represents the bid price. The expected utility of the seller who submits a limit order at a specified price P_S^L is $E[U] = \phi_S U (P_S^L - V_l - \tilde{\epsilon})$ where ϕ_S is the probability that the limit sell order is executed. The execution probability is given by:

$$\phi_S = \begin{cases} \frac{1}{2}\delta k + (1-k)\ (1-\frac{1}{2}\delta\gamma) & \text{if } U(0), \\ \frac{1}{2}(1-\delta)\, k & \text{if } U(P_S^L - V_l + H), \\ \frac{1}{2}k + \frac{1}{2}(1-k)\,\delta\gamma & \text{if } U(P_S^L - V_l - H), \end{cases}$$

In the next section, I derive the equilibrium bid and ask prices and their spread.

3 Equilibrium bid and ask prices

3.1 Concept of equilibrium

Equilibrium is defined as a set of mutual strategies such that each trader chooses an optimal strategy given the other traders' strategies. Traders can choose among the following strategies: a market order or a limit order and the bid price or ask price at which he or she places the limit order. The optimal bid price B^* and ask price A^* are determined to make the incoming trader submit a market order.

3.2 Equilibrium bid and ask price

The equilibrium prices in the limit order market are characterized only through the behavior of the uninformed trader. Because only the uninformed trader can choose either a market order or a limit order, they can only determine the specific price at which he or she posts the limit order. I then analyze the behavior of the current uninformed buyer to derive the ask price A^M at which the previous uninformed seller posts the limit order. The current uninformed buyer can choose either a market order or a limit order if there is at least one volume on the sell side of the limit order book. When he or she submits a market buy order, their utility is written as:

$$U_B^M = U(V_h - A^M),$$

where A^M is the ask price at which the previous uninformed seller posts the limit order. If the current uninformed buyer submits the limit order, I can then express the expected utility of the current uninformed buyer who places a limit buy order at the specified price B as:

$$E[U_B^L]=\beta_0U(0)+\beta_1U(V_h-H-B)$$
$$+\beta_2\{U(V_h+H-B)+U(V_h-H-B)\},$$

where:

$$\beta_0=\frac{1}{2}\delta(1-k)+k(1-\frac{1}{2}\delta\gamma),$$
$$\beta_1=\frac{1}{2}\delta(1-k)+\frac{1}{2}k\delta\gamma,$$
$$\beta_2=\frac{1}{2}(1-\delta)(1-k).$$

β_0 is the probability that the limit buy order expires. β_1 and β_2 are probabilities that the limit buy order is executed by the incoming informed seller and the incoming uninformed seller, respectively.

For the uninformed buyer, whether he or she submits a market order or a limit order depends on the ask price A^M that the previous uninformed seller posts. The current uninformed buyer submits a market order when U_B^M is greater than $E[U_B^L]$. The current uninformed buyer submits a limit order when U_B^M is less than $E[U_B^L]$. In the case that U_B^M equals $E[U_B^L]$, they choose either a market buy order or a limit buy order. For the previous uninformed seller, the lower ask price posted makes the current uninformed buyer submit a market order, but the previous uninformed trader's expected utility decreases by the ask price. In the case that a higher ask price is posted, the current uninformed buyer submits a limit order and the previous uninformed seller's expected utility is 0. Therefore the previous uninformed seller's expected utility in the case that U_B^M equals to $E[U_B^L]$ is higher than the other cases, $U_B^M<E[U_B^L]$ and $U_B^M>E[U_B^L]$. The previous uninformed seller posts the ask price A^M such that the current uninformed buyer's utility via a market buy order equals the current uninformed buyer's expected utility via a limit buy order, i.e.,

$$U_B^M=E[U_B^L].$$

Hence, the ask price A^M is given by:

$$A^M=\frac{1}{R}\log\left(\frac{\beta_0U(0)+\beta_1U(V_h-H-B)}{U(V_h)}+\frac{\beta_2\{U(V_h+H-B)+U(V_h-H-B)\})}{U(V_h)}\right). \tag{1}$$

If there is no volume on the sell side of the limit order book, then the uninformed buyer posts the limit order at the specified price B only. In this case, he or she posts B which makes the incoming uninformed seller's decision indifferent between a market order and a limit order; i.e., the incoming uninformed seller's utility via a market order equals the expected utility via a limit order. This is equivalent to the following case; i.e., whether the uninformed seller submits a market order or a limit order given bid price B^M. I then analyze the behavior of the uninformed seller. If there is at least one volume on the buy side of the limit order book, the uninformed seller can choose either a market order or a limit order and can post the specified price A. In the same way as the analysis of the uninformed buyer, the current uninformed seller submits either a market order or a limit order given the bid price B^M at which the previous uninformed buyer submits the limit order. In this case, the current uninformed seller's utility via a market order equals the current uninformed seller's expected utility via a limit order. The bid price B^M is then given by:

$$B^M = \frac{1}{R}\log\left(\frac{U^{-1}(V_l)}{\alpha_0 U(0)+\alpha_1 U(A-V_l-H)+\alpha_2\{U(A-V_l-H)+U(A-V_l+H)\}}\right), \qquad (2)$$

where:

$$\alpha_0 = \frac{1}{2}\delta k + (1-k)\left(1-\frac{1}{2}\delta\gamma\right),$$
$$\alpha_1 = \frac{1}{2}\delta k + \frac{1}{2}(1-k)\,\delta\gamma,$$
$$\alpha_2 = \frac{1}{2}(1-\delta)\,k.$$

α_0 is the probability that the limit sell order expires, α_1 and α_2 are the probabilities that the limit sell order is executed by the incoming informed buyer and the incoming uninformed buyer, respectively.

If there is no volume on the buy side of the limit order book, then the analysis of the uninformed buyer described above is applied. The equilibrium ask price A^* and bid price B^* are derived from (1) and (2), as all traders know the strategies of each other. As shown in proposition 1, the equilibrium bid price, ask price and their spread are composed of (i) the difference in valuation in the marketplace V_h and V_l, (ii) the adverse selection problem faced by an investor who decides to place a limit order, and (iii) the degree of risk aversion, which is represented by the risk aversion coefficient R.

Proposition 1. *The equilibrium bid and ask prices and their spread are given by:*

$$A^* = \frac{1}{R}\log\left(\frac{(b_1-a_1)+a_0 b_0 \exp\,(R(V_h-V_l))+\sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}}{2a_0\exp(-RV_l)}\right),$$

$$B^* = \frac{1}{R}\log\left(\frac{2b_0\exp(RV_h)}{(a_1-b_1)+a_0 b_0\exp(R(V_h-V_l))+\sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}}\right),$$

$$Spread^* = \frac{1}{R}\log\left(\frac{1}{2}\left[a_1+b_1+a_0 b_0\exp(R(V_h-V_l)+\sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}\right]\right),$$

where:

$$b_0 = -\beta_0 U(0),$$
$$b_1 = -[\beta_1 U(-H)+\beta_2\{U(+H)+U(-H)\}],$$
$$a_0 = -\alpha_0 U(0),$$
$$a_1 = -[\alpha_1 U(-H)+\alpha_2\{U(+H)+U(-H)\}],$$

and:

$f(a_0, a_1, b_0, b_1, R, V_h, V_l) = (a_1-b_1)^2+2a_0 b_0(a_1+b_1)\exp\,(R(V_h-V_l))+a_0^2 b_0^2\exp(2R(V_h-V_l))$.

The parameters $(a_0, a_1, b_0, b_1, R, V_h, V_l)$ *also satisfy follow equations from the condition that* $A^*>0$, $B^*>0$ *and* $Spread^*>0$.

$$a_1-b_1 < -a_0\exp(-RV_l)+b_0\exp(RV_h),$$
$$a_1+b_1 > 2-a_0 b_0\exp(R(V_h-V_l))-\sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}.$$

Proof. *See the appendix.* □

b_0 is the absolute expected utility when the limit buy order expires and b_1 is the absolute expected utility when the limit buy order is executed. Similarly, a_0 is the absolute expected utility when the limit sell order expires and a_1 is the absolute expected utility when the limit sell order is executed.

Because no useful result is obtained analytically at the equilibrium ask price A^*, bid price B^*, and their spread *Spread* *, I will describe an explicit numerical method for characterizing these equilibrium in section 4. In all the numerical analysis, I set the parameters as $V_h = 105$, $V_l = 95$, $H = 5$, $\delta = 0.75$, $R > 0.4$, and $\gamma = 0$ in order to compare the equilibrium ask, bid, and spread derived in Handa et al. (2003) with those derived in Proposition 1. These parameters satisfy the conditions in Proposition 1 and the condition that $B^* < V_l + H < A^*$ and $B^* < V_h - H < A^*$ for $0 < k < 1$ where k is the proportion of buyers in the market. The latter conditions mean the following. The unsatisfied cases with the condition for seller are:

$$\begin{cases} V_l + H < B^* & \cdots (a), \\ A^* < V_l + H & \cdots (b), \end{cases}$$

and the unsatisfied cases the condition for buyer are:

$$\begin{cases} V_h - H < B^* & \cdots (c), \\ A^* < V_h - H & \cdots (d). \end{cases}$$

In case (a), both the informed and uninformed seller obtain $\widetilde{W} = B^* - V_l - H$ certainly by submitting a market order, since it is assumed that all traders know parameters (V_h, V_l, k, δ, γ, R and the magnitude of H) ex ante. Then the uninformed seller does not submit the limit order; i.e., nobody sets the ask price. I exclude case (a) to analyze the equilibrium ask, bid prices and their spread in this paper. Incase (b), the informed seller obtains $\widetilde{W} = V_l + H - A^*$ certainly by switching to buyer. For analysis of the equilibrium, case (b) is excluded since it is assumed that the informed seller does not switch to be a buyer. Similarly, both the informed and uninformed buyer obtain $\widetilde{W} = V_h - H - A^*$ certainly by submitting a market order in case (d) and nobody sets the bid price, so this case is excluded. Case (c) is also excluded since the informed buyer obtains $\widetilde{W} = B^* - V_h + H$ certainly by switching to be a seller; i.e., $\gamma > 0$. The latter conditions, $B^* < V_l + H < A^*$ and $B^* < V_h - H < A^*$, represent that the equilibrium bid, ask prices and their spread exist and the informed buyer does not switch to be a seller, and vice versa; i.e., $\gamma = 0$. The cases of $k = 0$ and $k = 1$ are also excluded in the numerical analysis because nobody submits a market buy order when $k = 0$ and nobody submits a market sell order when $k = 1$.

4 A numerical analysis

In the analysis of the market, the equilibrium spread $A^* - B^*$ is an important measure characterizing the limit order market. In this section, I analyze the relationship between the equilibrium spread and the risk aversion coefficient of the uninformed trader's utility. I also show the relationship between the equilibrium spread and the proportion of buyers in the market in order to compare the equilibrium ask and bid prices and spread from Handa et al. (2003) with those derived in Proposition 1.

For comparison, I set the same parameters as Handa et al. (2003); i.e., $V_h = 105$, $V_l = 95$, $H = 5$ and $\delta = 0.75$. Since the uninformed trader has risk-averse utility in an Arrow-Pratt sense in this paper, it is assumed that the risk aversion parameter R is greater than 0.4. Although it is not obvious whether the parameters satisfy the condition that $B^* < V_l + H < A^*$ and $B^* < V_h - H < A^*$ are equivalent to $\gamma = 0$, I have checked these numerically and confirm that they are satisfied by the parameters.

Define the ask and bid commission as:

$$\textit{Ask Commission} = A^* - \bar{V},$$
$$\textit{Bid Commission} = \bar{V} - B^*,$$

where $\bar{V} = kV_h + (1-k)V_l$ is interpreted as the unconditional mean value of the risky asset. The equilibrium spread is defined as:

$$\textit{Spread} = A^* - B^*.$$

In order to analyze the relationship between the spread and the risk aversion coefficient R of the uninformed trader, see Figure2, which plots the spread for the case of an uninformed trader who has risk aversion coefficient R that is greater than 0.4. The proportion of buyers in the market k is set as 0.5. This means that the investors deal with buyers or sellers with equal probability.

Figure 2 shows that the increment of risk aversion makes the spread wider. Intuitively, this means that the uninformed buyer (seller) with more risk-averse utility requires a higher bid (ask) commission for trading risk when he or she submits the limit buy (sell) order. In particular, the risk whether the limit

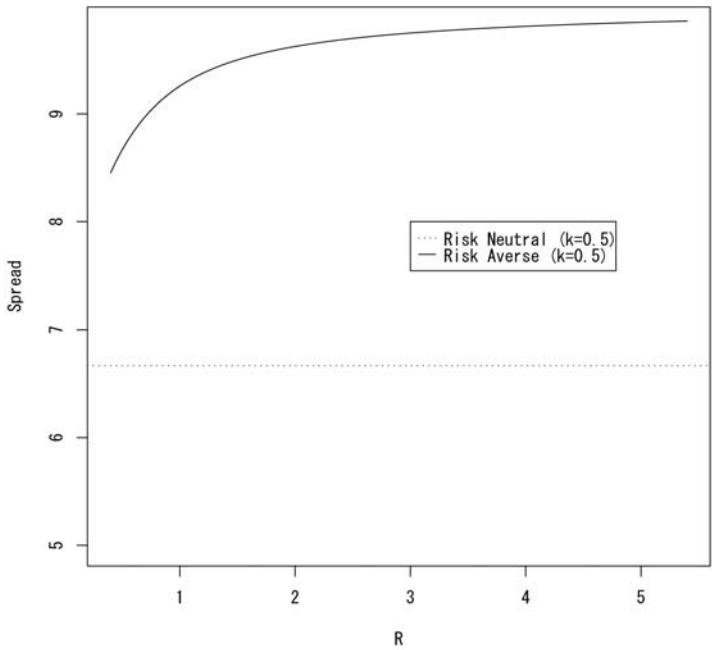

Figure 2 : Sprededs for risk-neutral and risk-averse utility.

order is executed is specific to the limit order market. This risk does not exist in the dealer market. The intuition is confirmed by Figure3, which describes the case of the uninformed trader who is risk averse.

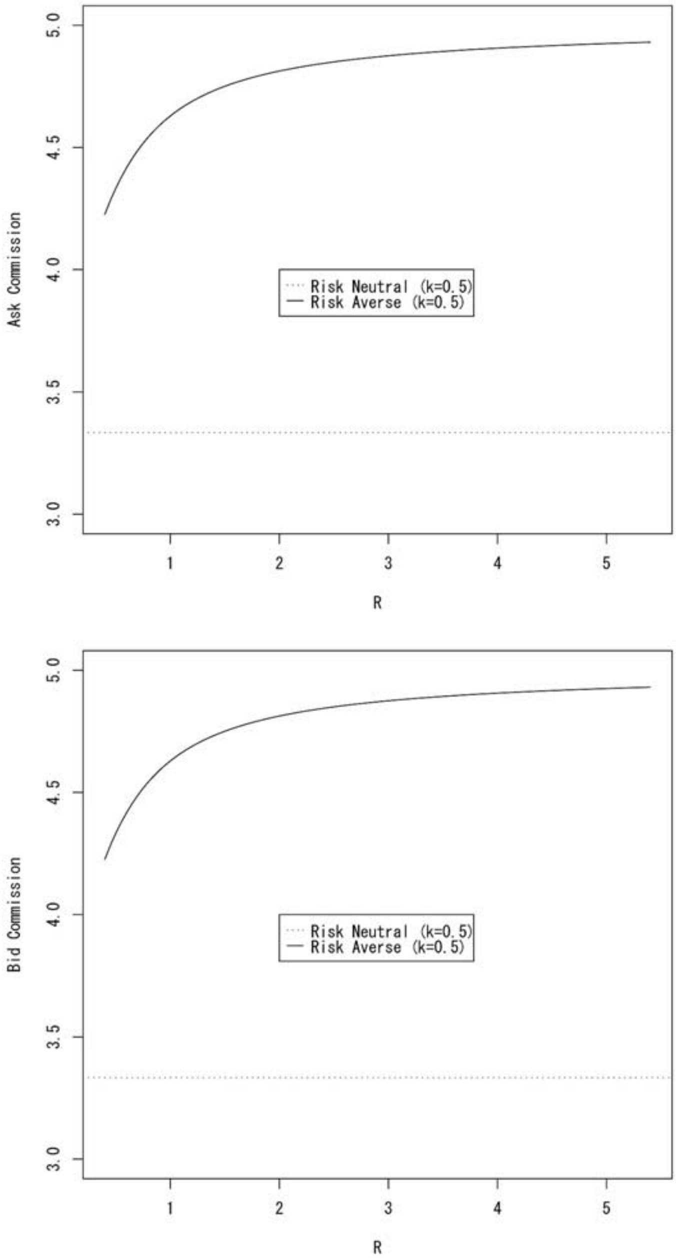

Figure 3 : Ask and did commissions for risk-nentual and risk-averse utility.

Figure 3 shows that the risk-averse uninformed trader requires higher ask and bid commissions than risk-neutral uninformed trader, and that both commissions are increasing with the increment in the degree of risk aversion R. Thus, the higher bid (ask) commission required by the uninformed buyer (seller) makes the spread wider.

The following Figure 4 shows the relationship between k and the equilibrium spread to make a comparison of spreads for risk-averse and risk-neutral uninformed traders.

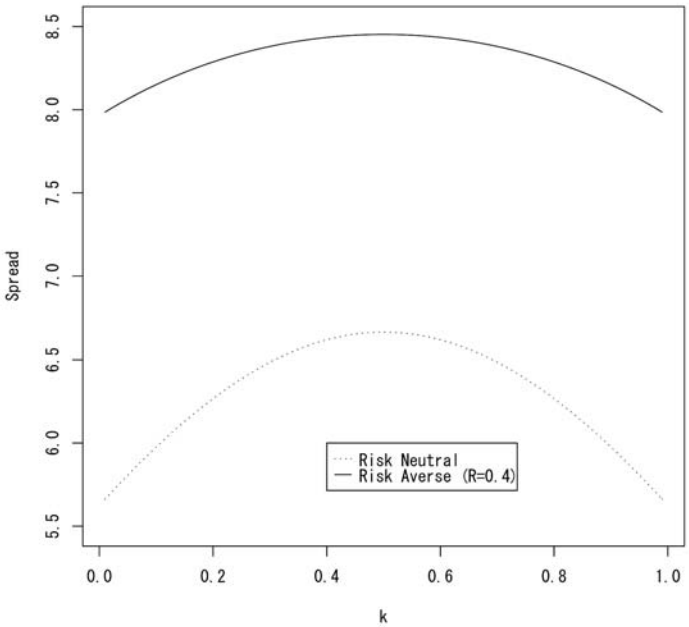

Figure 4 : Sprededs for risk-neutral and risk-averse utility over the proportion of buyers in the markets (k).

We see that the spread is higher and its form flatten if the uninformed trader has more risk averse utility for $0 < k < 1$. As is the case with Handa et al. (2003), the spread is lowest when k is located near 0 or 1 and is highest in the case when k is equal to 0.5. Figure5 describes the ask and bid commissions for the two types of trader; the uninformed trader with risk-neutral utility and risk-averse utility, respectively.

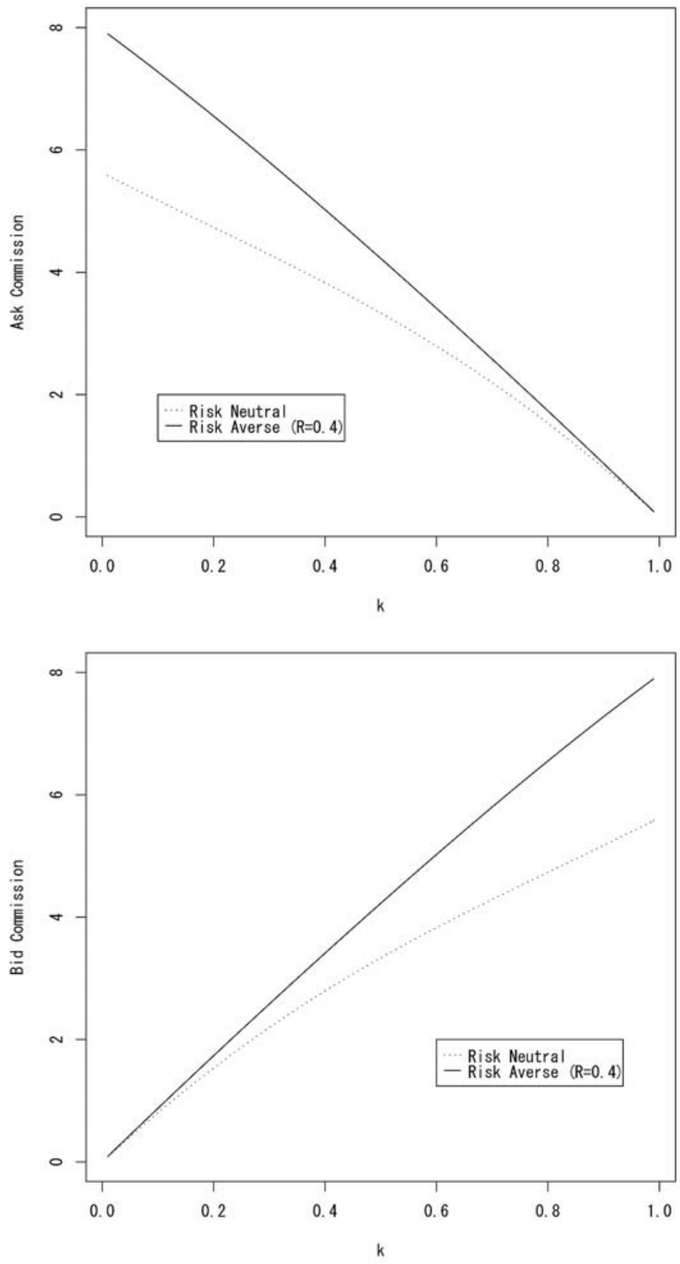

Figure 5 : Ask and did commissions for risk-neutral and risk-averse utilities over the proportion of buyers in the market (k).

It is certain that the ask commissions and the bid commissions are decreasing and increasing, respectively, with the increment in k, where k is the proportion of buyers in the market. This means that the probability of the limit sell (buy) order expiration is lower (higher) when k is closing to 1; i.e., the seller and buyer sides are thinner and thicker, respectively. The uninformed seller then requires a lower ask commission because his or her order is executed with certainty, and the uninformed buyer requires a higher bid commission for the risk that his or her limit buy order expires. Conversely, the uninformed seller requires a higher ask commission and the uninformed buyer a lower bid commission when k is closing to 0 because the seller side and the buyer side are thicker and thinner, respectively. The risk-averse uninformed trader requires a higher ask commission than the risk-neutral uninformed trader. The ask commissions which the former and latter respectively require grow closer and converge with the same value $\bar{V}$, which represents the unconditional mean value of the risky asset as k goes to 1. Similarly, the risk-averse uninformed trader requires a higher bid commission than the risk-neutral uninformed trader and their bid commissions converge with the same value $\bar{V}$ as k goes to 0. It is noted that both commissions take the same value $\bar{V}$ when the limit order is executed with certainty: that is, there is no risk of execution. If there is some risk of execution of the limit order, then the uninformed trader is more sensitive to risk than the risk-neutral uninformed trader. The different reactions towards risk make the spread change.

5 Conclusion

This paper examines the spread in the limit order market when the trader who submits the limit order has risk-averse utility in the Arrow-Pratt sense. It is shown that the spread is wider when he or she has more risk-averse utility as both ask and bid commissions increase with the increment in the risk aversion coefficient, representing the degree of the trader's sensitivity to risk. Thus, the trader with risk-averse utility is more sensitive to the risk that the limit order expires than the risk-neutral trader.

Ahn et al. (2002) show that both the spread in percentage terms and the number of transactions exhibit a U-shape form over time using intraday data on the Nikkei 225 stocks from the Tokyo Stock Exchange. That is, both indexes are higher at opening and closing. The higher number of transactions means that the proportion of buyers in the market is the same as the proportion of sellers. In this paper, the case is shown that the spread is at widest.

Handa et al. (2003) measure the spread over the proportion of buyers in the market using intraday data on stocks in the CAC40 index on the Paris Bourse. They conclude that the spread is wider or smaller every two months.

This finding is attributed to the fact that the degree of liquidity traders' risk aversion is changed during the period. The analysis also analyzes the equilibrium ask and bid prices and their spread when the risk-averse trader posts the limit order. The results show that the spread is strongly related to risk.

6 Appendix

Proof of Propositon 1.

In this paper, the buyer group and the seller group attach the risky asset to V_h and V_l, respectively. The proportion of the buyer group and the seller group in the market are defined as k and $1-k$, respectively. It is also assumed that there are two types of trader: informed traders and uninformed traders. The proportion of informed traders in the buyer (seller) group is δ. The informed trader can observe private information and he or she submits a market order only. Since the uninformed trader cannot observe the private information, then he or she chooses either a market order or a limit order. The informed trader has risk-neutral utility, while the uninformed trader has risk-averse utility in an Arrow-Pratt sense; i.e.:

$$U(\widetilde{W}) = -\exp(-R\widetilde{W}),$$

where $\widetilde{W}$ is wealth for the uninformed trader in trading. For the buyer who submits a market order, $\widetilde{W} = V_h - A^M$, where A^M is the ask price at which the previous seller posts the limit order. While $\widetilde{W} = V_h + \widetilde{\epsilon} - B$ for the buyer who submits a limit order where B is the specified price at which the current buyer posts the limit buy order and $\widetilde{\epsilon}$ is the private information which only the informed trader can observe.

$$\widetilde{\epsilon} = \begin{cases} +H & \textit{probability: } \dfrac{1}{2} \\ -H & \textit{probability: } \dfrac{1}{2} \end{cases}$$

Similarly for the seller who submits a market order, $\widetilde{W} = B^M - V_l$, where B^M is the bid price at which the previous buyer posts the limit order; while $\widetilde{W} = A - V_l - \widetilde{\epsilon}$. for the seller who posts a limit order where A is the specified price at which the current seller posts the limit order. It is assumed that $\widetilde{W}$ is normalized to 0 if the limit order submitted by traders expires.

From the above settings, the utility for the current uninformed buyer who submits the market buy order at price A^M is written as:

$$U_B^M = U(V_h - A^M).$$

If he or she posts a limit buy order at price B, then his or her expected utility is:,

$$\begin{aligned} E[U_B^L] = (1-k)\Big[& (1-\delta)\frac{1}{2}\{U(V_h+H-B) + U(V_h-H-B)\} \\ & + \delta\frac{1}{2}\{U(0) + U(V_h-H-B)\}\Big] + k\Big[(1-\delta)U(0) + \delta\Big\{\frac{1}{2}U(0) \\ & + \frac{1}{2}(\gamma U(V_h-H-B) + (1-\gamma)U(0))\Big\}\Big] \\ = {} & \beta_0 U(0) + \beta_1 U(V_h-H-B) + \beta_2\{U(V_h+H-B) + U(V_h-H-B)\}, \end{aligned}$$

where:

$$\beta_0 = \frac{1}{2}\delta(1-k) + k\left(1 - \frac{1}{2}\delta\gamma\right),$$

$$\beta_1 = \frac{1}{2}\delta(1-k) + \frac{1}{2}k\delta\gamma,$$

$$\beta_2 = \frac{1}{2}(1-\delta)(1-k).$$

β_0 is the probability that the limit buy order expires, β_1 and β_2 are the probabilities that the limit buy order is executed by the informed seller and the uninformed seller, respectively. The indifference between the current uninformed buyer's utility via market buy order and the current uninformed buyer's expected utility via limit buy order; i.e., $U_B^M = E[U_B^L]$ yields:

$$U(V_h - A^M) = \beta_0 U(0) + \beta_1 U(V_h - H - B) + \beta_2\{U(V_h + H - B) + U(V_h - H - B)\}. \qquad (3)$$

Therefore the price on the market buy order A^M is given as:

$$A^M = \frac{1}{R}\log\left(\frac{\beta_0 U(0) + \beta_1 U(V_h - H - B)}{U(V_h)} + \frac{\beta_2\{U(V_h + H - B) + U(V_h - H - B)\}}{U(V_h)}\right). \qquad (4)$$

On the other hand, the utility of the current uninformed seller who submits the market order at price B^M is:

$$U_S^M = U(B^M - V_l),$$

and the expected utility derived from the limit sell order at price A is:

$$\begin{aligned} E[U_S^L] &= k\Big[(1-\delta)\frac{1}{2}\{U(A - V_l - H) + U(A - V_l + H)\} \\ &\quad + \delta\frac{1}{2}\{U(A - V_l - H) + U(0)\}\Big] + (1-k)\Big[(1-\delta)U(0) + \delta\Big\{\frac{1}{2}U(0) \\ &\quad + \frac{1}{2}(\gamma U(A - V_l - H) + (1-\gamma)U(0))\Big\}\Big] \\ &= \alpha_0 U(0) + \alpha_1 U(A - V_l - H) + \alpha_2\{U(A - V_l - H) + U(A - V_l + H)\}, \end{aligned}$$

where:

$$\alpha_0 = \frac{1}{2}\delta k + (1-k)\left(1 - \frac{1}{2}\delta\gamma\right),$$

$$\alpha_1 = \frac{1}{2}\delta k + \frac{1}{2}(1-k)\delta\gamma,$$

$$\alpha_2 = \frac{1}{2}(1-\delta)k.$$

α_0 is the probability that the limit sell order expires, and α_1 and α_2 are the probabilities that the limit sell order is executed by the incoming informed buyer and the incoming uninformed buyer, respectively. The following equation is derived from the relation between the current uninformed seller's utility via the market sell order and the current uninformed seller's expected utility via the limit sell order; i.e.: $U_S^M = E[U_S^L]$,

$$U(B^M - V_l) = \alpha_0 U(0) + \alpha_1 U(A - V_l - H) + \alpha_2\{U(A - V_l - H) + U(A - V_l + H)\}. \qquad (5)$$

Thus, the price on the market buy order B^M is given by:

$$B^M = \frac{1}{R}\log\left(\frac{U^{-1}(V_l)}{\alpha_0 U(0) + \alpha_1 U(A - V_l - H) + \alpha_2\{U(A - V_l - H) + U(A - V_l + H)\}}\right). \qquad (6)$$

The above expressions (3), (4), (5) and (6), are provided for the equilibrium bid price B^* and ask price A^*. Firstly, the equations about A^* and B^* are derived from (3) and (5):

$$\exp(RA^*) = \beta_0 \frac{\mathrm{U}(0)}{U(V_h)} - [\beta_1 U(-H) + \beta_2\{U(+H) + U(-H)\}] \exp(RB^*),$$
$$= \frac{b_0}{\exp(-RV_h)} + b_1 \exp(RB^*). \tag{7}$$
$$\exp(-RB^*) = \alpha_0 U(0)\, U(V_l) - [\alpha_1 U(-H) + \alpha_2\{\mathrm{U}(+H) + U(-H)\}]\exp(-RA^*),$$
$$= a_0 \exp(-RV_l) + a_1 \exp(-RA^*), \tag{8}$$

where:

$$b_0 = -\beta^0 U(0),$$
$$b_1 = -[\beta_1 U(-H) + \beta_2\{U(+H) + U(-H)\}],$$
$$a_0 = -\alpha_0 U(0),$$
$$a_1 = -[\alpha_1 U(-H) + \alpha_2\{U(+H) + U(-H)\}].$$

b_0 and b_1 are the absolute expected utilities in the case that the limit buy order expires and is executed, respectively. Similarly, a_0 and a_1 are the absolute expected utilities in the case that the limit sell order expires and is executed, respectively. Secondly, the following equations are obtained by (7) and (8).

$$a_0 \exp(-R(V_h + V_l))\exp(RA^*) + (a_1 - b_1)\exp(-RV_h)$$
$$- a_0 b_0 \exp(-RV_l) - a_1 b_0 \exp(-RA^*) = 0$$
$$b_0 \exp(-RB^*) + (b_1 - a_1)\exp(-RV_h) - a_0 b_0 \exp(-RV_l)$$
$$- a_0 b_1 \exp(-R(V_h + V_l))\exp(RB^*) = 0$$

The exponential equilibrium of the ask price and the bid price follow on condition that exp (A^*) and $\exp(B^*)$ are greater than 0, respectively.

$$\exp(RA^*) = \frac{(b_1 - a_1)\exp(-RV_h) + a_0 b_0 \exp(-RV_l)}{2a_0 \exp(-R(V_h + V_l))}$$
$$+ \frac{\sqrt{\{(b_1 - a_1)\exp(-RV_h) + a_0 b_0 \exp(-RV_l)\}^2 + 4a_0 a_1 b_0 \exp(-R(V_h + V_l))}}{2a_0 \exp(-R(V_h + V_l))},$$
$$= \frac{(b_1 - a_1) + a_0 b_0 \exp(R(V_h - V_l)) + \sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}}{2a_0 \exp(-RV_l)}, \tag{9}$$
$$\exp(-RB^*) = \frac{(a_1 - b_1)\exp(-RV_h) + a_0 b_0 \exp(-RV_l)}{2b_0}$$
$$+ \frac{\sqrt{\{(a_1 - b_1)\exp(-RV_h) + a_0 b_0 \exp(-RV_l)\}^2 + 4a_0 b_0 b_1 \exp(-R(V_h + V_l))}}{2b_0},$$
$$= \frac{(a_1 - b_1) + a_0 b_0 \exp(R(V_h - V_l)) + \sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}}{2b_0 \exp(RV_h)}, \tag{10}$$

and the following equation is derived from (9), (10) and the condition that $A^* > 0$ and $B^* > 0$.

$$a_1 - b_1 = \left(k - \frac{1}{2}\right)\{(1 - \delta\gamma)\exp(RH) + (1 - \delta)\exp(-RH)\}$$
$$< -a_0 \exp(-RV_l) + b_0 \exp(RV_h)$$

Then exponential equilibrium spread is written as:

$$\exp(R(A^* - B^*)) = \frac{1}{2}\left[a_1 + b_1 + a_0 b_0 \exp(R(V_h - V_l)) + \sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)} \right], \qquad (11)$$

where:

$$f(a_0, a_1, b_0, b_1, R, V_h, V_l) = (a_1 - b_1)^2 + 2a_0 b_0 (a_1 + b_1) \exp(R(V_h - V_l)) + a_0^2 b_0^2 \exp(2R(V_h - V_l)),$$

and the condition that $A^* - B^* \geq 0$ gives:

$$\begin{aligned} a_1 + b_1 &= \frac{1}{2}\{(1+\delta\gamma)\exp(RH) + (1-\delta)\exp(-RH)\} \\ &> 2 - a_0 b_0 \exp(R(V_h - V_l)) - \sqrt{f(a_0, a_1, b_0, b_1, R, V_h, V_l)}. \end{aligned}$$

The equilibrium of the ask price A^*, the bid price B^*, and their spread $A^* - B^*$ are straightforwardly obtained from (9), (10) and (11).

□

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