Team production and constant share[*]

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Abstract

The author shows how restrictive it is to assume that a sharing rule is budget balancing in the context of team production. The class of budget balancing sharing rules can gain no better joint profit than a much thinner class of those, which the author defines as budget balancing constant shares. Constant share indicates a constant function of the team's income. Each level of the team's performance which is implementable with a budget balancing sharing rule is also implementable with a budget balancing constant share. Fixation of the sum of the shares causes such restrictedness. Two theorems proved or pointed out by Holmström (1982) are corollaries of the result.

1. Introduction

The author's main purpose is to show how restrictive it is to assume that a sharing rule is budget balancing in the context of team production. A sharing rule which economists usually define is some function from the team's income to individual income shares of the workers. Here the author defines the shares as a profile of the individual incomes' *ratio* to the team's total income.

Budget balancing sharing rules are the class of sharing rules afforded in the team production environment. A budget balancing sharing rule allocates the entirety of the team's income to its workers. It allows no surplus or deficits in its allocation.

The author shows that the class of budget balancing sharing rules can implement no more varieties of the workers' behavior than a much thinner class of those. Let the author define a notion of constant share. Constant share is a sharing rule which is a constant function of the team's income.[1] The main result asserts that each level of the team's performance which is implementable with a budget balancing sharing rule is also implementable with a budget balancing *constant share*. The result extends as far as a weighted sum of the shares is fixed.

Fixation of the sum of the shares causes such restrictedness. Generally the shares may change according to the team's income. The changes naturally strengthen or weaken incentives for the workers. Both of level and variation of the shares constitute the incentives together. However, the sum of the variations must be zero so that the sum of the shares themselves is constant. Then, if a profile of workers' behavior is implemantable by modifying the variations of a sharing rule, the

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1 Let the author remind the reader that the definition of the share is a ratio to the total income. Namely, a constant share has a fixed ratio of the workers' compensation.

same profile is implemeantable by some constant share. The workers' marginal revenue from their effort is the same between the former and the latter.

As long as one is interested in implementation of one particular level of the team's performance, income-dependent shift of the shares has no use to expand the set of implementable performance. Because, intuitively, whatever a sharing rule is, the total sum of incentives that can be allocated through the rule is fixed. Economists do not have to analyze all possible functional forms of a sharing rule when it is budget balancing. Especially, if they are interested in the team's performance, it suffices if they focus on the constant shares.

Two theorems proved or pointed out by Holmström (1982) are corollaries of the result.

First, any budget balancing sharing rule cannot implement the efficient outcome. The author's proof is much simpler than the original one.

Second, group penalty and group prize is necessary to implement the efficient outcome within the budget. The author analytically show that prohibition of such groupwise compensation technique forces a sharing rule to pay more than 100% of the team's income to implement the efficient outcome. The proof of this claim depends on qualitatively the same logic as that of the main result.

In the rest of the paper, first the author introduces the model and promptly shows the results in Section 2. Then the author refers to some related literature as qualifications and explains some implications in Section 3. Finally the conclusion of the paper appears with short comments on future research in Section 4.

2. Model and Results

There are N workers who contribute their nonnegative efforts $\mathbf{x} = (x_1, ..., x_N) \in R_+^N$ to their team. The effort individually costs according to individual cost functions $C_i : R_+ \rightarrow R_+$, $i \in \{1, ..., N\}$. A production function F determines an amount of monetary output with the profile of efforts given; $F : R_+^N \rightarrow R_+$. Here the author assumes that both of them are increasing and continuously twice differentiable, F is concave, and C is convex.

The workers share the output according to a sharing rule. It is a function from the volume of output to a profile of the workers' income *ratio* to the output; $\theta = \theta\ (F) : R_+ \rightarrow R_+^N$ and compensation to the worker i is $\theta_i(F(\mathbf{x}))\ F(\mathbf{x})$. The joint profit is $F(\mathbf{x}) - \sum_{i=1}^{N} C_i(x_i)$. Given θ, each worker maximizes her own individual profit by control of her effort.

The author focuses on a ratio of each worker's income and therefore has defined sharing rule as above. Note that the share is a ratio to the volume of output, not a level. When a sharing rule is budget balancing, it satisfies $\sum_{i=1}^{N} \theta_i(z) = 1,\ \forall z \in R_+$.

At first of the analysis, the author defines *a constant share*, which plays a key role.

Definition: A sharing rule θ is *a constant share* if $\forall z, z' \in R_+$, $\theta(z) = \theta(z')$.

Next the author offers a definition of equilibrium, equilibrium outcome and its efficiency.

Definition: $(\theta, \mathbf{x})$ is *an equilibrium* if each x_i maximizes the worker i's individual profit with the others' effort levels and θ as granted. $\mathbf{x}$ *is implemented by* θ if $(\theta, \mathbf{x})$ is an equilibrium.

Definition: An effort level profile $\mathbf{x}$ that is in an equilibrium is *the outcome* of that equilibrium.

Definition: An equilibrium outcome x is *efficient* if x maximizes the joint profit.

It is possible to show that the efficient x is unique, although the author omits the proof.

A constant share characterizes each equilibrium outcome, as a lemma below shows.

Lemma:

Any outcome x can be implemented by a constant share, which is

$$\theta^* = (\theta_1^*, \theta_2^*, ..., \theta_N^*), \theta_i^* = \frac{C_i'(x_i)}{F_i'(\mathbf{x})}, \text{ where } F_i'(\mathbf{x}) = \partial F(\mathbf{x})/\partial x_i. \quad (i \in \{1, 2, ..., N\}).$$

Proof: Given x, compute θ_i^* as above. Then θ^* satisfies $\theta_i^* \partial F(\mathbf{x})/\partial x_i - C_i'(x_i)$, $\forall i \in \{1, 2, ..., N\}$. It is the necessary first order conditions for individual profit maximization with constant share θ^*. The second order condition is satisfied due to concavity of the profit functions. Q.E.D.

The author names the constant share θ^* as *a corresponding constant share to x*. Finally the author defines a sharing rule with a fixed weighted sum.

Definition: *A sharing rule θ has fixed weighted sum* if

$\exists \mathbf{a} = (a_1, a_2, ..., a_N) \in R_{++}^N$ and $k > 0$, $\sum_{i=1}^N a_i \theta_i = k$.

a sharing rule has the fixed weighted sum for $\mathbf{a}$ if such $\mathbf{a}$ is given.

Note that when $a_i = 1\,(i \in \{1, 2, ..., N\})$ and $k = 1$, the fixed weighted sum condition is equivalent to the budget balancing conditon. Now the author offers the first theorem, which is a representation theorem for the class of sharing rules with some fixed weighted sum.

Theorem 1

Suppose that a sharing rule has the fixed weighted sum for some $\mathbf{a}$.

Each one equilibrium outcome of such a sharing rule is also implemented by a constant share, which satisfies the same fixed weighted sum condition to the original sharing rule.

Proof:

Suppose $(\theta, \mathbf{x})$ is an equilibrium. Let e_i be the ith unit vector with dimension of N. Δx_i denotes an increment of x_i. The author defines $F_i^+ = F(x + e_i \cdot \Delta x_i)$ and $C_i^+ = C_i(x_i + \Delta x_i)$.

Definition: $\{\Delta x\} = \{\Delta x = (\Delta x_1, ..., \Delta x_N) \mid F_i^+ = F_j^+ \forall i, j \in \{1, ..., N\}\}$.

Lemma: $\forall \Delta x, \Delta x \gg 0, \Delta x = 0$, or $0 \gg \Delta x$. (Proof is omitted.)

$\forall \Delta x \gg 0, \forall i,\ \{\theta(F_i^+) F_i^+ - C_i^+\} - \{\theta(F)F - C_i\} \le 0$, because it is an equilibrium.

It is equivalent to $\{\theta_i(F_i^+) - \theta_i(F)\} F + \theta_i(F_i^+) \{F_i^+ - F\} - \{C_i^+ - C_i\} \le 0$.

Multiplying $\dfrac{F}{F_i^+ - F}$ on both sides,

$$\frac{\theta_i(F_i^+) - \theta_i(F)}{F_i^+ - F} F^2 + \theta_i(F_i^+) F - \frac{F}{F_i^+ - F}\{C_i^+ - C_i\} \le 0.$$

Taking weighted sum with weight of $\mathbf{a}$,

$$\sum_{i=1}^N a_i \frac{\theta_i(F_i^+) - \theta_i(F)}{F_i^+ - F} F^2 + \sum_{i=1}^N a_i \theta_i(F_i^+) F - \sum_{i=1}^N a_i \frac{F}{F_i^+ - F}\{C_i^+ - C_i\} \le 0.$$

Due to the fixed weighted sum condition, $kF - \sum_{i=1}^N a_i \dfrac{F}{F_i^+ - F}\{C_i^+ - C_i\} \le 0$.

Taking limit of $\Delta x \to 0$, $\quad kF - \sum_{i=1}^{N} a_i \frac{F}{F_i'} C_i' \leq 0 \Leftrightarrow k - \sum_{i=1}^{N} a_i \frac{1}{F_i'} C_i' \leq 0.$

Similarly, $\forall \Delta x \ll 0$, $\quad k - \sum_{i=1}^{N} a_i \frac{1}{F_i'} C_i' \geq 0$, so that $k - \sum_{i=1}^{N} a_i \frac{1}{F_i'} C_i' = 0.$

Corresponding constant share θ_i^* satisfies $\sum_{i=1}^{N} a_i \theta_i^* = k \quad$ and $\quad \theta_i^* F_i' - C_i' = 0.$

Q.E.D.

As seen in the proof above, if the original sharing rule has multiple equilibria, for each equilibrium outcome there exists one corresponding constant share independently. Now the author simply proves the inefficiency theorem by Holmström (1982).

Lemma:

Among variety of the constant shares, there is unique one that implements the efficient out-come, which is $\theta^* = (1, 1, ..., 1)$.

Proof: When the joint profit is maximized, following condition must be satisfied: $\partial F(x)/\partial x_i = C_i'(x_i)$, $\forall i \in \{1, 2, ..., N\}$. Therefore $\forall i$, $\theta_i^* = 1$. Q.E.D.

Corollary:

As far as a sharing rule has the fixed weighted sum for some **a**, that sharing rule can implement the efficient outcome only if $k = \sum_{i=1}^{N} a_i$.

When the simple sum of the shares is fixed, i.e. each a_i is equal to 1, the efficienct out-come is implementable only when $\sum_{i=1}^{N} \theta_i = N$. Clearly the budget balancing condition implies impossibility of implementing the efficient outcome. This is the Holmström's theorem.

What is the main cause of the inefficiency? The author asserts that it is impossibility of group penalty and prize, in other words, the restriction that all of individual shares never can rise or fall simultaneously. A theorem below supports the author's assertion.

Theorem 2:

Suppose that any sharing rule θ *has to satisfy following conditions.*

(1) *Right hand and left hand differentiation exist.*

(2) $\forall z \in R_+$, $\exists \epsilon > 0$,

$\forall z' \in (z - \epsilon, z + \epsilon) \cap R_+$, $\exists i\ \theta_i(z) - \theta_i(z') > 0 \Leftrightarrow \exists j\ \theta_j(z) - \theta_j(z') < 0.$

(3) $\forall z \in R_+$,

$$\exists i\ \lim_{z' \to z+} \frac{\theta_i(z) - \theta_i(z')}{z - z'} = \infty \Leftrightarrow \exists\ j\ \lim_{z' \to z+} \frac{\theta_j(z) - \theta_j(z')}{z - z'} = -\infty,$$

$$\exists i\ \lim_{z' \to z-} \frac{\theta_i(z) - \theta_i(z')}{z - z'} = \infty \Leftrightarrow \exists\ j\ \lim_{z' \to z-} \frac{\theta_j(z) - \theta_j(z')}{z - z'} = -\infty.$$

Consider an equilibrium where the efficient outcome is implemented. There, the realized share among workers is excessively paying, in a sense that at least one worker's share is more than or equal to one. If no worker's share is strictly more than one, then each worker's share has to be exactly one.

Conditions (2) and (3) is the prohibition of group penalty. Condition (1) is a restriction on sharing rules and necessary for the proof. The theorem 2 implies that when we prohibit group penalty and prize, we cannot implement the efficient outcome within the budget.

Proof:

Step 1: A jump point or an inflection point of θ cannot be an equilibrium output.

Consider a case where right hand limit of $\dfrac{\theta_i(F_i^+)-\theta_i(F)}{F_i^+-F}$ for some i has the infinite value (Proof for left hand case is similar). Then from condition (3), there exists $j \in \{1, 2, ..., N\}$ such that $\lim\limits_{F_j^+\to F} \dfrac{\theta_j(F_j^+)-\theta_j(F)}{F_j^+-F} = \infty$ and thus $\dfrac{\theta_j(F_j^+)-\theta_j(F)}{F_j^+-F}F^2+\theta_j(F_j^+)F-\dfrac{F}{F_j^+-F}\{C_j^+-C_j\} > 0$ for some F_j^+, which implies that the worker j will deviate from the equilibrium.

Step 2: the proof of the excessive payment.

Suppose that in an equilibrium, $x = x^*$. Δx is defined in the same way to theorem 1.

Take an ϵ as defined in (2).

$\forall \Delta x \gg 0$ s.t. $F_i^+ - F < \epsilon$, $\forall i$,

$$\frac{\theta_i(F_i^+)-\theta_i(F)}{F_i^+-F}F^2+\theta_i(F_i^+)F-\frac{F}{F_i^+-F}\{C_i^+-C_i\} \le 0.$$

Let us define

$$d^+ \equiv \begin{cases} -\dfrac{\sum_{i=1}^N \max(0, \theta_i(F_i^+)-\theta_i(F))}{\sum_{i=1}^N \min(0, \theta_i(F_i^+)-\theta_i(F))} & \text{if } \theta(F_i^+)-\theta(F) \ne 0, \\ 1 & \text{if } \theta(F_i^+)-\theta(F) = 0, \end{cases}$$

and construct following $\mathbf{a}^+$:

$$\mathbf{a}^+ \equiv (a_1^+, a_2^+, ..., a_N^+),\ a_i^+ = \begin{cases} 1 & \text{if } \theta_i(F_i^+)-\theta_i(F) \ge 0, \\ d^+ & \text{if } \theta_i(F_i^+)-\theta_i(F) < 0. \end{cases}$$

Then, taking weighted sum over i with $\mathbf{a}^+$ as the weight,

$$\sum_{i=1}^N a_i^+\theta_i(F_i^+) - \sum_{i=1}^N a_i^+ \frac{1}{F_i^+-F}\{C_i^+ - C_i\} \le 0.$$

Set $a'^+ = \lim\limits_{F_i^+\to F} a^+$. Existence of the limit is given by condition (1) and the step 1. Then

$$\sum_{i=1}^N a_i'^+\theta_i(F) - \sum_{i=1}^N a_i'^+ \frac{1}{F_i'} C_i' \le 0 \Leftrightarrow \sum_{i=1}^N a_i'^+\theta_i(F) - \sum_{i=1}^N a_i'^+\theta_i^* \le 0.$$

Similarly, $\exists a'^-$, ϵ, $\forall \Delta x \ll 0$ s.t. $F_i^+ - F < \epsilon$, $\forall i$, $\sum_{i=1}^N a_i'^-\theta_i(F) - \sum_{i=1}^N a_i'^-\theta_i^* \ge 0$.

There exists $\alpha \in [0, 1]$, $\mathbf{a} = \alpha a'^- + (1-\alpha)a'^+$, $\sum_{i=1}^N a_i\theta_i(F) - \sum_{i=1}^N a_i\theta_i^* = 0$.

By construction, $\mathbf{a}$ is strictly positive. The efficiency implies $\theta^* = (1, 1, ..., 1)$.

Therefore, $\sum_{i=1}^N a_i\theta_i(F) = \sum_{i=1}^N a_i$.

$\theta_i(F) = 1\,(i \in \{1, 2, ..., N\})$ naturally satisfies the condition above. And if it is not the case, there has to be $i \in \{1, 2, ..., N\}$ such that $\theta_i(F)$ is strictly greater than 1. Q.E.D.

If the group penalty is available, an ideal sharing rule exists, as in Holmström (1982):

$$\forall i \in \{1, 2, ..., N\},\ \theta_i = \begin{cases} \theta_i^{**} & \text{if } F(\mathbf{x}) \geq F(\mathbf{x}^*), \\ 0 & \text{otherwise,} \end{cases} \quad \text{where } \theta_i^{**} = \frac{C_i'(\mathbf{x}^*)}{F_i'(\mathbf{x}^*)}.$$

The realized share balances the budget.

3. Literatures and Implications

A group penalty or prize is possible when there exists a powerful principal who can claim residual. Eswaran and Kotwal (1984) points out that use of such group payment scheme "introduces the potential for moral hazard" by the principal. If the principal can make side contract with an arbitrary worker, the principal will make the worker shirk intentionally for small amount of payment. Then the principal can acquire positive residual, rather than zero.

The risk neutrality of the workers is crucial for our result. Rasmusen (1982) shows that there exists a budget balancing sharing rule which implements the efficient outcome when the workers are risk averse. In that sharing rule, when the level of output is lower than the targeted level, a scapegoat is randomly chosen among the workers and he/she is punished by a negative reward. His/her fine is shared by the other workers. If the workers are sufficiently risk averse, or if the fine can be sufficiently heavy, the risk of this lottery plays a role of threat for *all* workers, and the efficient outcome is implemented in the equilibrium.

On one hand, any constant share cannot substitute such sharing rule. It manages to make group penalty according to the level of output, while any constant share cannot make the group penalty. On the other hand, the sharing rule slightly violates the assumption here. It depends on the negative reward, in other words, the workers' liabilities. Without positive liabilities, it requires rather high (in the original model, infinite) degree of the workers' risk aversion. In the context of practical labour contract, rewards for the workers may often be restricted so as to be non-negative.

The results are negative theorems. However, the theorem 1 is useful in applications, because when a researcher is interested only in the team's performance, it suffices if he/she analyzes only the class of constant shares. It makes the analysis splendidly easy.

4. Conclusion

Based on a generic model of team production with certainty, the author has shown the following two things. (1) The fixed weighted sum condition on the workers' income ratio is equivalent to limiting sharing rules to be constant shares which satisfy the same condition, and (2) including the budget balancing condition, prohibition of group penalty induces sharing rule to pay excessively for implementation of the fficient outcome. Focus on the constant share as a characterizing tool is the key to this study. The author has not touched uncertainty cases and dedicates them to future research. Also, specifying the production technology can be another agenda.

References

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