Note on Utility Maximization Problem via Duality Method:

How to Derive the Candidate for the Dual Formulation

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Abstract

In this paper, we consider the utility maximization problem in the incomplete market models, and derive the candidate for its dual problem. These arguments have already been given by Chen et al. (2008) or Rogers (2003), we also provide the way to derive the candidate of the dual formula and the optimal portfolio value at the terminal.

Keywords: Utility Maximization Problem, Duality, Incomplete Market Model

1 Introduction

We consider the derivation of the dual problem for the utility maximization problem with random endowments (*Primal Problem*). Such a problem is not only to. nd the optimal investment strategy, but also useful to derive the utility indi. erence price (Hodges and Neuberger (1989)). Rogers already has provided an intuitive scheme to derive the candidate dual problem for the no-endowment problem, the so-called classical utility maximization problem. These arguments have already been provided by Chen et al. (2008) in a non-traded asset model (Henderson and Hobson (2009), Musiela and Zariphopoulou (2004)). This paper however extends their incomplete model to more general incomplete market model and includes the stochastic volatility model. We also provide a way to derive the candidate of the dual formula and the optimal portfolio value at the terminal. Furthermore, we explicitly give the dual relation with the primal problem.

2 Financial Market

Let us consider the following. nancial market. There exists one risky asset (typically a stock) and one risk-free asset (typically a bank account). Initially, for a bank account B, we assume that the risk-free rate is r for time horizon $[0, T]$: this value process is then expressed as

$$\mathrm{d}B(s) = rB(s)\,\mathrm{d}s.$$

Next, we set the stock price process in an incomplete market. The uncertainty in this market is characterized by a probability space $(\Omega, \mathcal{F}, P)$. We then introduce a two-dimensional standard

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Brownian motion denoted by $W=(W_1, W_2)$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, where $\mathcal{F}_t$ is the filtration generated by $(W(s); 0\leq s\leq t)$ and satisfies the usual conditions. Under the above settings, the stock price process and another state price process are defined as follows

$$\mathrm{d}S(s) = S(s)\{\mu(s, Y(s))\mathrm{d}s + \sigma(s, Y(s))\mathrm{d}W_1(s)\} \tag{2.1}$$

where Y is a certain state variable driven by

$$\mathrm{d}Y(s) = a(s, Y(s))\mathrm{d}s + b(s, Y(s))\{\rho\mathrm{d}W_1(s) + \sqrt{1-\rho^2}\,\mathrm{d}W_2(s)\} \tag{2.2}$$

for $0\leq s\leq T$ and $-1<\rho<1$.

Assumption 2.1. *Coefficients* $\mu(\cdot, Y(\cdot))$, $\sigma(\cdot, Y(\cdot))$, $a(\cdot, Y(\cdot))$ *and* $b(\cdot, Y(\cdot))$ *in* (2.1) *and* (2.2) *are smooth, and bounded above and below away from zero.*

We suppose that there is a European-type claim whose payo. function is $g(T):=g(T, S(T), Y(T))$ at the maturity T. The above settings have considered both the stochastic volatility model (Sircar and Zariphopoulou (2005)) and non-traded asset model.

Next we construct the wealth process. Let denote $H(s)$ amounts of the stock by self-financing rule at time $s\in[t, T]$. The wealth process is then given by

$$X^{H,x}(T) = x + \int_0^T H(s)\mathrm{d}S(s) \tag{2.3}$$

where $X(0)=x$. Alternatively $X^{H,x}$ solves to

$$\mathrm{d}X^{H,x}(s) = H(s)S(s)\{\mu(s, Y(s))\mathrm{d}s + \sigma(s, Y(s))\mathrm{d}W_1(s)\}. \tag{2.4}$$

For simplicity, we denote by X the wealth process instead of $X^{H,x}$. We denote by $\chi(x, k)$ the family of nonnegative wealth processes with $X(0)=x$, i. e.,

$$\chi(x, k) = \{X : X(0) = x, X(T) + kg(T) \geq 0\}.$$

For later discussion, we introduce the exponential local martingale Z^ν sucn that

$$Z^\nu(t) = \exp\left(-\int_0^t (\lambda(s)\mathrm{d}W_1(s) + \nu(s)\mathrm{d}W_2(s)) - \frac{1}{2}\int_0^t (Z^2(s) + \nu^2(s))\mathrm{d}s\right) \tag{2.5}$$

where $Z(s)=(\mu(s, Y(s))-r)/\sigma(s, Y(s))$ and ν is supposed to satisfy

$$E\left[\frac{1}{2}\int_0^T \nu^2(s)\mathrm{d}s\right]<\infty. \tag{2.6}$$

From Assumption 2.1 and (2.6), Novikov condition is satisfied, i.e.,

$$E\left[\frac{1}{2}\int_0^T(\lambda^2(s) + \nu^2(s))\mathrm{d}s\right]<\infty,$$

and this leads that Z^ν is a martingale, then we can define the equivalent martingale measure Q with P as follows

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = Z^\nu(t)$$

for $0\leq t\leq T$.

3 Dual Formula

We consider the following utility maximization problem (primal problem) with k random endowments (alternatively, *claim*).

$$u(x,k)=\sup_{X\in\mathcal{X}(x,k)} E[U(X(T)+kg(T))] \tag{3.1}$$

where U is the utility function (typically the exponential utility or the power utility).

Set an auxiliary process Z by $Z(s):=y\beta(s)Z^{\nu}(s)$ for $0\leq s\leq T$ where $\beta(s)=1/B(s)$, which solves to

$$\mathrm{d}Z(s)=Z(s)\{-r\mathrm{d}s+\lambda(s)\mathrm{d}W_1(s)+\nu(s)\mathrm{d}W_2(s)\}$$

with $Z(0)=y$. Then, the next proposition gives the candidate of the dual formula and the duality relationship with the primal problem, and also optimal $X(T)$.

Proposition 3.1. *Let us denote the dual problem v as*

$$v(y)=\inf_{\nu} E[\tilde{U}(Z(T))+Z(T)kg(T)] \tag{3.2}$$

for $y>0$. *Then we have*

$$u(x)=\inf_{y>0}[v(y)+xy],$$

and the optimal $\hat{X}$ *is form of*

$$\hat{X}(T)=I(Z(T))-kg(T)$$

where I is the inverse of U'.

Proof. By the integral by parts, we have

$$\begin{aligned}\int_0^T Z(s)\mathrm{d}X(s)&=Z(T)X(T)-Z(0)X(0)-\int_0^T X(s)\mathrm{d}Z(s)-\langle Z,X\rangle(T)\\&=Z(T)X(T)-Z(0)X(0)\\&\quad-\int_0^T X(s)Z(s)\{-r\mathrm{d}s+\lambda(s)\mathrm{d}W_1(s)+\nu\,\mathrm{d}W_2(s)\}\\&\quad-\int_0^T Z(s)\lambda(s)H(s)S(s)\sigma(s,Y(s))\mathrm{d}s.\end{aligned} \tag{3.3}$$

On the other hand, the wealth process $\mathrm{d}X$ is given by (2.3). Substituting (2.3) into $\int_0^T Z(s)\,\mathrm{d}X(s)$ gives

$$\int_0^T Z(s)\mathrm{d}X(s)=\int_0^T Z(s)\{(rX(s)+H(s)S(s)(\mu(s,Y(s))-r))\mathrm{d}s+H(s)S(s)\sigma(s,Y(s))\mathrm{d}W_1(s)\}. \tag{3.4}$$

Taking expectation (3.3) and (3.4) give

$$I_1:=E\left[\int_0^T Z(s)\mathrm{d}X(s)\right]$$

$$= E\left[Z(T)X(T) - Z(0)X(0) - \int_0^T Z(s)\,\{-rX(s) + \lambda(s)H(s)S(s)\sigma(s, Y(s))\}\,\mathrm{d}s\right],$$

$$I_2 := E\left[\int_0^T Z(s)\,\mathrm{d}X(s)\right] = E\left[\int_0^T Z(s)\,(rX(s) + H(s)S(s)\,(\mu(s, Y(s)) - r))\,\mathrm{d}s\right],$$

respectively. Therefore, we have

$$\begin{aligned} 0 &= I_1 - I_2 \\ &= E\left[Z(T)X(T) - Z(0)X(0) - \int_0^T Z(s)\{\lambda(s)H(s)S(s)\sigma(s, Y(s)) + H(s)S(s)\,(\mu(s, Y(s)) - r)\}\,\mathrm{d}s\right] \\ &= E[Z(T)X(T) - Z(0)X(0)]. \end{aligned} \tag{3.5}$$

By (3.5), we obtain Lagrangian L as the upper bound of (3.1):

$$L(Z) = \max_{x \geq 0} E\tilde{L}(H, X, Z) \tag{3.6}$$

where

$$\tilde{L}(H, X, Z) = U(X(T) + kg(T)) - Z(T)X(T) + Z(0)X(0).$$

That is, it holds

$$u(x) \leq L(Z) \tag{3.7}$$

for any $x, y > 0$ and v .

The first order condition (FOC) to achieve the maximum value $\hat{L}(Z)$ are given by

$$\hat{X}(T) = I(Z(T)) - kg(T) \tag{3.8}$$

as a solution of $\partial\tilde{L}/\partial X(T) = 0$, where $I(\cdot)$ is the inverse function of $U'(\cdot)$. Substituting (3.8) into (3.6), we have maximized value $\hat{L}(Z)$ of (3.6)

$$\hat{L}(Z) = E[\tilde{U}(Z(T)) + Z(T)kg(T) + Z(0)X(0)] \tag{3.9}$$

where $\tilde{U}(\cdot)$ is the convex dual function defined by

$$\tilde{U}(y) = \sup_x [U(x) - xy].$$

From (3.7) and (3.9), we obtain the dual formula

$$\begin{aligned} u(x) &= \inf_{y>0,\, v} E[\tilde{U}(Z(T)) + Z(T)kg(T) + Z(0)X(0)] \\ &= \inf_{y>0} [\inf_v E[\tilde{U}(Z(T)) + Z(T)kg(T)] + Z(0)X(0)] \\ &= \inf_{y>0} [v(y) + xy]. \end{aligned}$$

Setting $k = 0$ in Proposition 3.1, we have the following relationship.

Corollary 3.1. *Let us consider the utility maximization problem u with no random endowment and* *its dual problem v as follows*:

$$u(x) = \sup_{X(T) \in \chi(x, 0)} E[U(X(T))] \quad \textit{for } x > 0,$$

$$v(y) = \inf_{v} E[\tilde{U}(Z(T))] \quad \text{for } y > 0.$$

We have then the following dual relation

$$u(x) = \inf_{y>0} [v(y) + xy],$$

and the optimal $\hat{X}$ is form of

$$\hat{X}(T) = I(Z(T)).$$

Results of Corollary 3.1 coincides with the results of Karatzas et al. (1991), Kramkov and Schachermayer (1999) (Exercise 3A, in Rogers (2003)) and Pham (2008).

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