Exponential Hedging vs Mean-Variance Hedging:
Numerical Examples in an Incomplete Market

Kazuhiro Takino[*]

Abstract

In this article, we consider the exponential hedging and the mean-variance hedging in the basis risk model. The basis risk model is a typical example of the in the incomplete market model. The basis risk model describes the market model in which the underlying asset of the contingent claim is not traded in the financial market. We compare the hedge performances for both the mean-variance hedging and the exponential hedging by simulating hedge errors. We further demonstrate the exponential hedging for two different initial hedge costs and risk-aversions respectively. These demonstrations give the optimal hedging cost (i.e., utility indifference price) which leads the well performance in the exponential hedging, and also verify the relation between the risk-averse and the performance of the exponential hedging.

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1 Introduction

In this article, we consider the hedging problems for the European type contingent claim taking into account of the position of the claim in the basis risk model. We then demonstrate the hedge performance for both mean-variance hedging and exponential hedging.

The basis risk model (Davis (2006), Duffie and Richardson (1991), Henderson (2002), Monoyios (2004), Musiela and Zariphopoulou (2004) and Schweizer (1992)) is a typical example of the incomplete market model, which includes the model that the underlying asset of the contingent claim is not traded in the financial market. The pricing models for the weather derivative or the derivative written on the market index are recognized as one of the basis risk model for instance. In the complete market (the typical example is Black-Scholes model), any contingent claims are replicated with traded assets. On the other hand, the value of the claim is not surly attained with traded assets in the incomplete market model. This means that the seller of the claim is exposed to have the hedge error risk, so she/he wants to control it with her/his preference. The exponential hedging and the mean-variance hedging have independently developed in context of finding the optimal hedging strategy for the contingent claim in the incomplete market model. The exponential hedging approach is usually formulated to maximize the expected utility for the amounts of which the hedge portfolio exceeds the claim payoff. The mean-variance hedging, on the other hand, is a hedging criteria to minimize the hedge error measured by $\boldsymbol{L}_2$-norm, and does not take into account the investor's preference for the risk. This problem is solved by using the projection in the Hilbert

[*] takino@nucba.ac.jp

space (c.f., Schweizer (2001)). The significant difference of the both approaches is whether it includes the risk preference of the market participant or not. The exponential hedging reflects the investor's attitude for the risk since it is based on the utility maximization with the exponential utility. The exponential hedging also has been developed in context of the utility indifference pricing with the exponential utility (e.g., Delbaen et al., (2002), Henderson (2002), Ilhan et al., (2004), Mania and Schweizer (2005), Monoyios (2004, 2008) and Musiela and Zariphopoulou (2004)). Mania and Shcweizer (2005) showed that the exponential hedging strategy with the utility indifference price converges to the strategy of the mean-variance hedging when the risk-averse coefficient goes to zero. Ilhan et al., (2004) also summarized that the utility indifference price converges to the expectation of the discounted payoff under the minimal martingale measure when the risk-averse coefficient goes to zero. This value coincides with the mean-variance hedging cost in our basis-risk model.

Finally we review previous literatures related to our study before closing this section. Monoyios (2004) demonstrated the comparison of the exponential hedging with the utility indifference price and a naive strategy used the BS-Delta in ad-hoc. We use his implementation scheme to our study since he derived the asymptotic expansion formula for the utility indifference price and the exponential hedging. On the other hand, we compare the exponential hedging and the mean-variance hedging. It goes without saying that the mean-variance hedging strategy provides the minimum hedging error among various hedging strategies include the exponential hedging. By comparing hedge performances for two hedging strategies, however, we verify the advantages/disadvantages for these hedging strategies. In fact, there are few papers corresponding to such a comparison. Furthermore, we simulate the exponential hedging strategies for two difference hedging costs. The one is the utility indifference price, the another is the mean-variance hedging cost. This implementation has never been considered in previous researches to our knowledge. So this is a contribution of the present study. These demonstrations provides the following results: The mean-variance hedging takes the minimum hedging cost and minimum hedging error risk, while the possibility of the success hedge is weaker than the exponential hedging. The exponential hedging, on the other hand, higher performance compared with the mean-variance hedging in the point of that the hedge portfolio exceeds the claim payoff called ***hedge error***. Finally, the exponential hedging exerts its advantages for the derivative hedging with the utility indifference price as the initial hedging cost.

We also demonstrate the hedging performance for the exponential hedging strategies for different risk-aversions. From that, we observe that the risk-aversion affects the distribution of the hedge error in the exponential hedging as expected and especially improves the shortfall risk.

The rest of paper is organized as follows: In Section 2, we set up the financial market model. We construct the basis risk model. In Section 3, we solve the mean-variance hedging problem. In Section 4, we derive the utility indifference price and the exponential hedging respectively. In Section 5, we demonstrate the both mean-variance hedging and exponential hedging in numerical way. We then discuss the characteristic for several hedging schemes from those numerical results. Finally, we conclude this study in Section 6.

2 Model

In this section, we set the financial market model so-called basis risk model. We also construct the equivalent martingale measure.

2.1 Financial Market Model

We consider the basis risk model (or non-traded asset model). That is, there are one risky asset S (typically a stock), one risk-free asset B (typically a bank account) with zero risk-free rate and one state level Y which is supposed to be not traded in the financial market. For instance, as to the weather derivative case, Y corresponds to a weather index such as the average temperature. Let us set the value process for above instruments. The uncertainty in this market is characterized by a probability space $(\Omega, \mathcal{F}, P)$. We then introduce a two-dimensional standard Brownian motion denoted by $W = (W_1, W^{\perp})$ on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, where $\mathcal{F}_t$ is the filtration generated by $(W(t); 0 \leqslant s \leqslant t)$ and satisfies the usual conditions.

The value process of the risk-free asset B is

$$\mathrm{d}B(t) = 0$$

with $B(0) = 1$ and risk-free rate is 0. The stock price process S and the state level Y are supposedto be driven by

$$\begin{aligned} \mathrm{d}S(t) &= S(t)\{\mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}W_1(t)\} \\ \mathrm{d}Y(t) &= Y(t)\{\mu_2 \mathrm{d}t + \sigma_2 \mathrm{d}W_2(t)\} \end{aligned}$$

for $0 \leqslant t \leqslant T$, where $W_2 := \rho W_1 + \sqrt{1-\rho^2}\, W^{\perp}$ $(-1 < \rho < 1)$, μ_i and σ_i $(i = 1, 2)$ are constants.

We would price a European-type claim whose payoff function is denoted by $H := H(t, Y(T))$ at maturity T. In the numerical simulation, we consider the put option as an example. This allows us to use an asymptotic expansion introduced by Monoyios (2004). We assume $H \in \mathcal{L}_2(P)$, where $\mathcal{L}_2(P)$ is a space of square integrable random variables, i.e.,

$$\mathcal{L}_2(P) = \{X \text{ is r.v.} : E|X|^2 < \infty\}.$$

We use European put option in the numerical example. The put option ensures to verify the asymptotic expansion of the exponential hedging and the indifference price as demonstrated in Monoyios (2004).

The hedging strategies are constructed by the self-financing rule. That is, the hedge portfolio value $X(t)$ $(0 \leqslant t \leqslant T)$ is driven by

$$\mathrm{d}X(t) = \kappa(t)\mathrm{d}S(t), \quad X(0) = x$$

with the hedging strategy κ which is amount held in the stock S. $X(0)$ corresponds the initial hedging cost, it is assigned to the utility indifference price in the exponential hedging as explained in the following section. Now we give the mathematical condition of κ, i.e., admissible policy.

Definition 2.1 (Admissible). *The portfolio strategy* κ *is admissible if it satisfies*

$$E\left[\int_0^T \kappa^2(t)S^2(t)\mathrm{d}t\right] < \infty.$$

Therefore, we denote by $\mathcal{A}$ *the set of all admissible policies* κ.

2.2 Equivalent Martingale Measure

For two-dimensional predictable process $\lambda = (\lambda_1, \lambda_2)^\top$ we introduce

$$\begin{aligned} Z(t) &= \varepsilon(-\lambda \cdot W) \\ &:= \exp\left(-\frac{1}{2}\int_0^t -(\lambda_1(s)^2 + \lambda_2(s)^2)\mathrm{d}s - \int_0^t (\lambda_1(s)\mathrm{d}W_1(s) + \lambda_2(s)\mathrm{d}W^\perp(s))\right). \end{aligned} \tag{2.1}$$

We assume that $\theta := \lambda_1(t) \equiv (\mu_1 - r)/\sigma_1$ and $\lambda = (\lambda_1, \lambda_2)^\top$ satisfies Novikov condition

$$E\left[e^{\frac{1}{2}\int_0^t (\lambda_1(s)^2 + \lambda_2(s)^2)\mathrm{d}s}\right] < \infty.$$

Then Z is a martingale under P. Z is also a solution of

$$\mathrm{d}Z(t) = -Z(t)\,(\lambda_1(s)\mathrm{d}W_1(s) + \lambda_2(s)\mathrm{d}W^\perp(s)), \quad Z(0) = 1.$$

Defining an equivalent probability measure P^λ by

$$\frac{\mathrm{d}P^\lambda}{\mathrm{d}P} = Z(T),$$

then P^λ is an equivalent martingale measure. Under P^λ measure, the risky asset price discounted with the risk-free asset becomes a martingale. Set $\lambda = (\theta, 0)^\top$, P^λ then yields the minimal martingale measure denoted by Q. The density process of Q is also given by $Z(t) = \varepsilon(-\theta W_1)$. Define $\widetilde{W} = (\widetilde{W}_1, \widetilde{W}^\perp)$ by

$$\widetilde{W}_1(t) = W_1(t) + \int_0^t \theta \mathrm{d}s, \quad \widetilde{W}^\perp(t) = W^\perp(t),$$

then, from Girsanov's theorem, $\widetilde{W}$ is two-dimensional Brownian motion under Q.

3 Mean-Variance Hedging

In the present section, we consider the mean-variance hedging strategy for multiple units of claim. The result argued in this section is a basis for the main theorem. The purpose of the mean-variance hedging is to find a hedge portfolio strategy $\kappa(t) \in \mathcal{A}$ $(0 \leqslant t \leqslant T)$ with the initial cost C (constant) to minimize the hedge error with $\mathcal{L}_2$-norm. Define so-called gain process G by $G(t) = \int_0^t \kappa(s)\mathrm{d}S(s)$, then the value process of the hedge portfolio is represented by

$$X(t) = C + G(T)$$

since the initial hedging cost $X(0)$ is $C^{(k)}$. Our purpose is formally to find κ such that

$$\min_{\kappa \in \mathcal{A}} \left(E[H - C - G(T)]^2\right)^{\frac{1}{2}}. \tag{3.1}$$

3.1 Variance-Optimal Martingale Measure

The mean-variance hedging strategy is constructed with Galtchouk-Kunita-Watanabe decomposition of the claim H under so-called Variance Optimal Martingale Measure (VOMM). We denote VOMM by P^*. In particular, the initial hedging cost of H is given by the expected value of the discounted payoff of H under VOMM. We would like to recommend the reader to refer Schweizer (2001) and Pham (2009) more detail explanations for the mean-variance hedging. We thus first need to specify VOMM P^*. We define the VOMM according to Pham (2009).

Definition 3.1 (Variance-Optimal Martingale Measure: VOMM). *The equivalent martingale measure P^λ is the variance-optimal martingale measure if it solves to*

$$\inf_{P^\lambda \in \mathcal{M}_e} E\Big[\frac{\mathrm{d}P^\lambda}{\mathrm{d}P}\Big]^2.$$

It is easy to find the VOMM for our basis-risk model.

Proposition 3.1 (Variance Optimal Martingale Meausre). *The variance-optimal martingale measure P^* is given by*

$$\frac{dP^*}{dP} = Z^*(t) := \varepsilon(-\theta W_1)$$

in our financial market model introduced in previous section.

Proof. Under the real world measure P, the discount risky asset price $\hat{S}$ is represented by using the martingale term M and the finite variation A

$$\hat{S}(t) = \hat{S}(0) + M(t) + A(t)$$

where $M(t) = \int_0^t \sigma_1 X(s)\mathrm{d}W_1(s)$ and $A(t) = \int_0^t \eta(s)\mathrm{d}\langle M\rangle(s)$ with $\eta(t) = \theta/\sigma_1 X(t)$. Then the mean-variance tradeoff process J defined by

$$J(t) = \int_0^t \eta(s)^2 \mathrm{d}\langle M\rangle(s)$$

is then deterministic. Therefore, Lemma 4.7 in Schweizer (2001) completes the proof. □

From Proposition 3.1, Z^* solves to

$$\mathrm{d}Z^*(s) = -\theta Z^*(s)\mathrm{d}\widetilde{W}_1(s) = -\theta Z^*(s)\frac{1}{\sigma_1 S(s)}\mathrm{d}S(s) = -\frac{\theta}{\sigma_1 S(s)}\mathrm{d}S(s) =: \frac{\zeta(s)}{Z^*(s)}\mathrm{d}S(s) \qquad (3.2)$$

where $\zeta(s) = -\theta Z^*(s)/\sigma_1 S(s) = -\eta(s)Z^*(s)$.

Remark 3.1. *The variance optimal martingale measure P^* in our model coincides with the minimal martingale measure Q. $\widetilde{W}$ given in the last of Section 2.2, is also two-dimensional Brownian motion under P^*.*

3.2 Mean-Variance Hedging

In this section we give the mean-variance hedging strategy. To this end, we first derive the perfect hedging strategy for the claim H under VOMM P^* by reference to Heath et al., (2001).

The value processes X and Y are respectively driven by

$$\mathrm{d}\hat{S}(t) = \sigma\hat{S}(t)\mathrm{d}\widetilde{W}_1(t), \tag{3.3}$$
$$\mathrm{d}Y(t) = (\mu_2 - \rho\theta\sigma_2)Y(t)\mathrm{d}t + \sigma_2 Y(t)\mathrm{d}\widetilde{W}_2(t) \tag{3.4}$$

under P^*, where $\mathrm{d}\widetilde{W}_2 = \rho\mathrm{d}\widetilde{W}_1 + \sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}^{\perp}$. The Galtchouk-Kunita-Watanabe decomposition for $H \in L^2(P)$ under P^* is then given by

$$H = E^*[H\,|\mathcal{F}_0] + \int_0^T \xi^{H,P^*}(s)\mathrm{d}\hat{S}(s) + L^{H,P^*}(T) =: V^{H,P^*}(T) \tag{3.5}$$

with

$$V^{H,P^*}(t) := E^*[H\,|\mathcal{F}_0] + \int_0^t \xi^{H,P^*}(s)\mathrm{d}\hat{S}(s) + L^{H,P^*}(t), \quad 0 \leqslant t \leqslant T. \tag{3.6}$$

Both of $\int_0^t \xi^{H,P^*}(s)\mathrm{d}\hat{S}(s)$ and L^{H,P^*} are martingales under P^*, $\int_0^t \xi^{H,P^*}(s)\mathrm{d}\hat{S}(s)$ is orthogonal to L^{H,P^*} under P^*.

Now we solve ξ^{H,P^*} and $L^{H,P*}$. Put

$$v^*(t, \hat{S}(t), Y(t)) = E^*[H\,|\mathcal{F}_t]$$

from Markov property. Feynman-Kac formula yields that $v^*(t,\hat{S}(t), Y(t))$ is a solution of

$$\partial_t v^* + (\mu_2 - \rho\theta\sigma_2)y\partial_y v^* + \frac{1}{2}\sigma_1^2 x^2 \partial_{SS} v^* + \sigma_1\sigma_2\hat{S}_y\partial_{Sy} v^* + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} v^* = 0 \tag{3.7}$$

with $v^*(T, \hat{S}, y) = H$, where $\partial_z f = \partial f(z)/\partial z$. On the other hand, by Ito's formula, we have

$$\begin{aligned} \mathrm{d}v^* = &\left(\partial_t v^* + (\mu_2 - \rho\theta\sigma_2)y\partial_y v^* + \frac{1}{2}\sigma_1^2\hat{S}^2\partial_{SS}v^* + \sigma_1\sigma_2\hat{S}_y\partial_{Sy}v^* + \frac{1}{2}\sigma_2^2 y^2\partial_{yy}v^*\right)\mathrm{d}t \\ &+ \partial_S v^*\mathrm{d}\hat{S}(t) + \sigma_2 y\partial_y v^*\mathrm{d}\widetilde{W}_2(t). \end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.8) we obtain

$$\mathrm{d}v^*(t, \hat{S}(t), Y(t)) = \partial_S v^*(t, \hat{S}(t), Y(t))\mathrm{d}\hat{S}(t) + \sigma_2 y\partial_y v^*(t, \hat{S}(t), Y(t))\mathrm{d}\widetilde{W}_2(t), \tag{3.9}$$

then it holds

$$v^*(t, \hat{S}(t), Y(t)) = v^*(0, \hat{S}, y) + \int_0^t \partial_S v^*(s, \hat{S}(s), Y(s))\mathrm{d}\hat{S}(s) + \int_0^t \sigma_2 Y(s)\partial_y v^*(s, \hat{S}(s), Y(s)\mathrm{d}\widetilde{W}_2(s) \tag{3.10}$$

By comparison between (3.6) and (3.10), we have ξ^{H,P^*} and L^{H,P^*} as

$$\xi^{H,P^*}(t) = \partial_S v^*(t, \hat{S}(t), Y(t)) + \frac{\rho\sigma_2 Y(t)}{\sigma_1\hat{S}(t)}\partial_y v^*(t, \hat{S}(t), Y(t)), \tag{3.11}$$
$$L^{H,P^*}(t) = \int_0^t \sigma_2\sqrt{1-\rho^2}\,Y(s)\partial_y v^*(s, \hat{S}(s), Y(t))\mathrm{d}\widetilde{W}^{\perp}(s). \tag{3.12}$$

Theorem 3.1. *The mean-variance hedging strategy* (C, κ^*) *for* $H \in \mathcal{L}_2(P)$ *is given by*

$$C = V^{H,P^*}(0) = E^*[H]$$
$$\kappa^*(t) = \xi^{H,P^*}(t) + \eta(t)(V^{H,P^*}(t) - C - G^{mvh}(t))$$

for $0 \leqslant t \leqslant T$, *where* $\eta(t) = \theta/\sigma_1 S(t)$ *and* G^{mvh} *is the gain process for the mean-variance hedging strategy* κ^*, *i.e.,* $G^{mvh}(t) = \int_0^t \kappa^*(s)d\hat{S}(s)$.

Proof. The assumption of $H \in \mathcal{L}_2$ gives the initial cost of the mean-variance hedging strategy as follows

$$C = E^*[H]$$

from the first assertion of Theorem 3.1. Next, we verify that κ^* is the mean-variance hedging strategy. To do this, we set

$$I(t) = E[(V^{H,P*}(t) - C - G^{mvh}(t))\, G(t)]$$

for $0 \leqslant t \leqslant T$. From Lemma 1 in Duffie and Richardson (1991), the optimality of κ^* is equivalent to satisfy

$$I(T) = 0. \tag{3.13}$$

Defining $D(t) := V^{H,P*}(t) - C - G^{mvh}(t)$ with $D(0) = 0$, leads

$$\begin{aligned} \mathrm{d}D(t) &= \mathrm{d}V^{H,P*}(t) - \mathrm{d}G^{mvh}(t) \\ &= \xi^{H,P*}(t)\mathrm{d}\hat{S}(t) + \mathrm{d}L^{H,P*} - \kappa^*(t)\mathrm{d}X(t) \\ &= (\xi^{H,P*}(t) - \kappa^*(t))\mathrm{d}\hat{S}(t) + \mathrm{d}L^{H,P*}(t) \\ &= -\eta(t)D(t)\mathrm{d}(M(t) + A(t)) + \mathrm{d}L^{H,P*}(t). \end{aligned}$$

From Ito's formula and the orthogonal relation between $\int \kappa \mathrm{d}\hat{S}$ and $L^{H,P*}$ we have

$$\begin{aligned} \mathrm{d}(D(t)G(t)) &= D(t)\mathrm{d}G(t) + V(t)\mathrm{d}D(t) + \mathrm{d}\langle D, G\rangle(t) \\ &= D(t)\kappa(t)\mathrm{d}A(t) - \eta(t)D(t)G(t)\mathrm{d}A(t) - D(t)\kappa(t)\mathrm{d}A(t) + (martingales) \\ &= -\eta(t)D(t)G(t)\mathrm{d}A(t) + (martingales) \\ &= -\theta^2 D(t)G(t)\mathrm{d}t + (martingales). \end{aligned}$$

From Fubini's theorem, we obtain

$$\begin{aligned} I(t) = E[D(t)G(t)] &= E\left[\int_0^t \theta^2 D(s)G(s)\mathrm{d}s\right] \\ &= \int_0^t (-\theta^2)E[D(s)G(s)]\mathrm{d}s \\ &= -\int_0^t \theta^2 I(s)\mathrm{d}s \end{aligned}$$

since θ^2 is deterministic. So it holds that

$$\mathrm{d}I(t) = -\theta^2 I(s)\mathrm{d}t, \quad 0 \leqslant t \leqslant T.$$

This yields

$$I(t) = 0, \quad 0 \leqslant t \leqslant T$$

with $I(0) = 0$, and shows (3.13). □

4 Exponential Hedging and Utility Indifference Price

In this section, we construct the exponential hedging strategy based on the utility indifference price for the claim. This is already demonstrated by Monoyios (2004, 2008) for a unit of claim, the indifference price is used as the initial hedging cost.

4.1 Utility Indifference Price with Exponential Utility

In this section, we derive the utility indifference price as the initial hedging cost in the exponential hedging. The indifference price derived by solving two distinct utility maximization problems. The one is so-called Merton's problem to maximize the expected utility from terminal portfolio value, the other is one from terminal portfolio value equipped with claims. Delbaen et al., (2002) and Monoyios (2004, 2008) considered the later problem as the exponential hedging, in particular Monoyios derived a hedging strategy for the claim.

In order to derive the utility indifference price, we set utility maximization problems. The market participant has an exponential utility with the risk averse coefficient $\gamma>0$ as follows:

$$U(x) = -\frac{1}{\gamma}e^{-\gamma x}$$

for $x>0$. Set the portfolio strategy $\pi := \kappa S$, then π means the money amount held in the stock. We use π as an optimizer of the following utility maximization problems in this section for convenience. The portfolio value process is thus given by

$$\mathrm{d}X(t) = \frac{\pi(t)}{S(t)}\mathrm{d}S(t).$$

We denote the set of all admissible policies π for all $\kappa \in \mathcal{A}$ by $\mathcal{A}'$

The problem to maximize the expected utility from terminal portfolio value is given by

$$u_0(t, x) = \sup_{\pi \in \mathcal{A}'} E_t[U(X(T))]$$

where E_t denotes the expectation conditioned with the market information $\mathcal{F}_t$ up to time t. On the other hand, the problem to maximize the expected utility from terminal portfolio value with k claims is represented by

$$u(t, x, y) = \sup_{\pi \in \mathcal{A}'} E_t[U(X(T) - H)].$$

This is the value function for the exponential heding (Delbaen et al., (2002)).

Definition 4.1 (Utility Indifference Price). *The utility indifference price p(t; k) for k claims at time t is a solution of*

$$u_0(t, x) = u(t, x + p(t), y). \tag{4.1}$$

Since the investor receives the premium p at the initial time, so p in Definition 4.1 implies the seller's price.

The basis risk model permits the explicit solutions for u0 and u respectively with the exponential utility, this leads explicit representation of p (c.f., Musiela and Zariphopoulou (2004)).

Proposition 4.1. *The utility indifference price p(t) at time t for k units of claim is*

$$p(t) = \frac{1}{\gamma(1-\rho^2)} \ln \tilde{E}_t[e^{\gamma(1-\rho^2)H}] \tag{4.2}$$

where $\tilde{E}$ denotes the expectation under Q-measure.

Proof. See Musiela and Zariphopoulou (2004). □

4.2 Exponential Hedging

The exponential hedging has been considered by Delbaen et al., (2002), the value function of the hedging problem arises in the utility indifference price approach with the exponential utility, i.e., $u(t, x, y)$. Furthermore, Monoyios (2004, 2008) derived the closed formula for the hedging strategy by using an asymptotic expansion scheme. The hedging strategies demonstrated by Monoyios (2004, 2008) use the utility indifference price as the initial hedging cost. We thus apply Monoyios's works to our hedging problem.

Proposition 4.2. *The exponential hedging strategy δ held in the stock is given by*

$$\delta(t) = \frac{\rho\sigma_2 y}{\sigma_1 S}\partial_y p(t). \tag{4.3}$$

Proof. See Monoyios (2004). □

4.2.1 Asymptotic Expansion of Exponential Hedging Strategy

Let us derive an asymptotic expansion of the exponential hedging strategy, i. e., δ in (4.3). Since we have no closed formula for $p(t)$ in (4.2), it is convenience to use the asymptotic formula of $p(t)$ to obtain a closed formula of the hedging strategy. Monoyios (2004) respectively derived an asymptotic expansion of $p(t)$ and $\delta(t)$, and we use his scheme to our study.

Proposition 4.3 (Monoyios (2004)). *The utility indifference price $p(t)$ the claim with the exponential utility is represented by*

$$\begin{aligned} p(t; k) = m_1(t) &+ \frac{1}{2}\gamma\varepsilon^2(m_2(t) - m_1^2(t)) + \frac{1}{3!}\gamma^2\varepsilon^4(m_3(t) - 3m_1(t)m_2(t) + 2m_1^3(t)) \\ &+ \frac{1}{4!}\gamma^3\varepsilon^6(m_4 - 3m_2^2(t) - 4m_1(t)m_3(t) + 12m_1^2(t)m_2(t) - 6m_1^4(t)) + O(\varepsilon^8), \end{aligned} \tag{4.4}$$

where $\varepsilon = \sqrt{1-\rho^2}$ and $m_i(t) = \tilde{E}_t[H^i]$ $(i = 1, 2, ...)$, if the parameters satisfy

$$\tilde{E}[e^{\gamma\varepsilon^2 H}] \leqslant 2. \tag{4.5}$$

The closed form of $m_i(t) = \tilde{E}_t[H^i]$ $(i = 1, 2, ...)$ is given in Section 6.1 in Monoyios (2004). From Proposition 4.3, we obtain a closed formula of the exponential hedging strategy (4.3) by calculating the first derivative of (4.4). From (4.4), we have

$$\begin{aligned} \partial_y p(t) = \partial_y m_1(t) &+ \frac{1}{2}\gamma\varepsilon^2(\partial_y m_2(t) - 2m_1(t)\partial_y m_1(t)) \\ &+ \frac{1}{3!}\gamma^2\varepsilon^4(\partial_y m_3(t) - 3\partial_y m_1(t)m_2(t) - 3m_1\partial_y m_2(t) + 6m_1^2(t)\partial_y m_1(t)) \\ &+ \frac{1}{4!}\gamma^3\varepsilon^6(\partial_y m_4 - 6m_2(t)\partial_y m_2(t) - 4m_3(t)\partial_y m_1(t) - 4m_1(t)\partial_y m_3(t) \\ &+ 24m_1(t)m_2(t)\partial_y m_1(t) + 12m_1^2(t)\partial_y m_2(t) - 24m_1^3(t)\partial_y m_1(t))) + O(\varepsilon^8). \end{aligned} \tag{4.6}$$

5 Numerical Example and Main Result

In this section, we demonstrate each hedging strategies and compare the hedge performances for those hedging scheme. We investigate the hedge performance by simulating the hedge error defined by

$$X(T) - H(T, Y(T))$$

where $X(T)$ is the terminal value of the hedge portfolio and $H(T, Y(T))$ denotes the payoff of the claim. We use Monte-Carlo Simulation to demonstrate the hedging strategies, and the parameters used in this example are presented in Table 1.

Table 1: Parameters used in the numerical examples.

Parameter	Value
$S(0)$	100
μ_1	0.01
σ_1	0.25
$Y(0)$	100
μ_2	0.12
σ_2	0.30
K	100
T	1.0

We consider the three hedging strategies:

1. the mean-variance hedging denoted 'MVH'
2. the exponential hedging with the mean-variance hedging cost denoted 'Exp. Naive'
3. the exponential hedging with the utility indifference price denoted 'Exp.' for $\gamma = 0.005, 0.01$.

5.1 Comparison between Mean-Variance Hedging and Exponential Hedging

Table 2–4 summarize the statistics for the results of the simulations with varying ρ with 100,000 simulation times. From Table 2, the utility indifference price requires more the initial hedging cost than the mean-variance hedging. Table 3 exactly shows the performance of each hedging strategies. From table, the exponential hedging with the utility indifference price has the most high-performance among three strategies. The value function of the exponential hedging problem implies the reason for this result. The utility level increases if the hedge portfolio value exceeds the claim payout, this leads that the exponential hedging tends to have the hedge performance same as the one of the superhedging strategy. The same result will be reported as *success hedge ratio* in the followings. Table 4 shows very natural outcomes because of the objective function of the mean-variance hedging. If we measure the hedge error risk with the standard deviation of the hedge error, the mean-variance hedging results in the smallest hedge error risk. We note that the exponential hedging with mean-variance hedging cost provides worst performance (Table 3, 4).

Finally, we also implement the hedge performance. The sample path is arose 10,000 times with

Table 2: The initial hedge cost. For the ‘Exp. Naive’ strategy, the initial hedging cost is used the mean-variance hedging cost. The column of ‘Exp.’ shows the utility indifference price.

			Exp.	
ρ	MVH	Exp. Naive	$\gamma = 0.005$	$\gamma = 0.01$
− 0.75	4.7309	←	4.8558	4.9653
− 0.5	5.5034	←	5.7079	5.9314
− 0.25	6.3596	←	6.6750	7.0023
0	7.3013	←	7.6810	8.0784
0.25	8.3294	←	8.7167	9.1294
0.5	9.4436	←	9.7946	10.1554
0.75	10.6405	←	10.8626	11.0881

Table 3: The average of the hedge error.

			Exp.	
ρ	MVH	Exp. Naive	$\gamma = 0.005$	$\gamma = 0.01$
− 0.75	− 0.4745	− 0.4024	− 0.2865	− 0.1613
− 0.5	− 0.2331	− 0.0972	0.0375	0.3251
− 0.25	− 0.1199	− 0.0203	0.2235	0.6077
0	0.0137	− 0.0048	0.3819	0.7792
0.25	− 0.0127	− 0.1437	0.3150	0.6827
0.5	− 0.0086	− 0.2711	0.1099	0.4100
0.75	− 0.2323	− 0.5959	− 0.3449	− 0.1427

Table 4: The standard deviation of the hedge error.

			Exp.	
ρ	MVH	Exp. Naive	$\gamma = 0.005$	$\gamma = 0.01$
− 0.75	7.7194	7.5992	7.5053	7.5945
− 0.5	9.5986	10.2937	9.7792	10.2860
− 0.25	10.9539	11.7409	11.3108	11.7466
0	11.5624	12.0164	11.9986	11.9986
0.25	11.5798	12.1159	11.9214	12.1129
0.5	10.6222	11.0927	10.9901	11.1047
0.75	8.6604	8.9458	8.9072	8.9600

$\rho = 0.75$ and $\gamma = 0.005, 0.01$. This experience verifies the performance of the exponential hedging with the indifference price. To this end, we define the success hedge ratio by

$$\text{Success Hedge Ratio} = \frac{\#(X(T) \geqslant H(T, Y(T))}{10000}.$$

The results are shown in Figure 1 and Table 5. The results are similar to the above, we focus on the success hedge ratio in the bottom of the table. The hedging strategy which provides the most high-

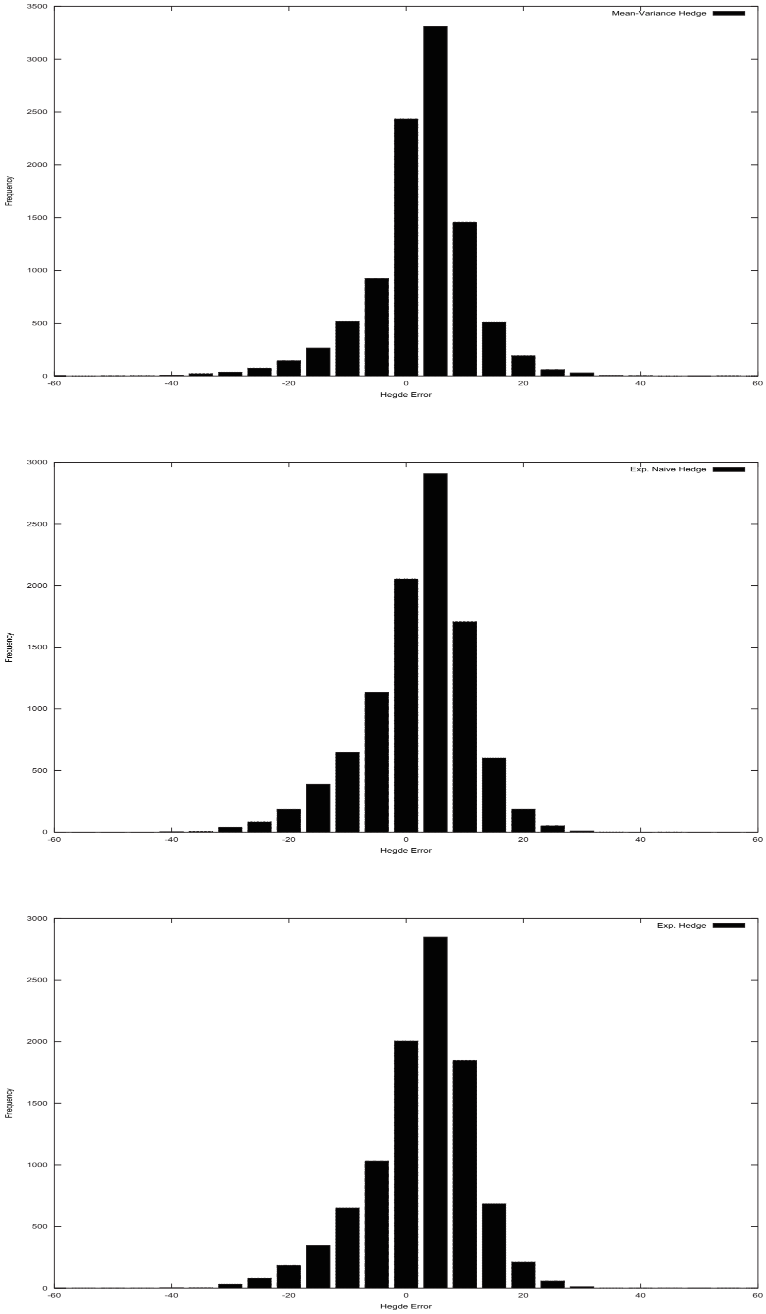

Figure 1: The distribution of the hedge error. The graphs respectively show the histogram for the mean-variance hedging, the exponential hedging with the naive cost and the exponential hedging with the utility indifference cost (with $\gamma = 0.01$) from above. Each graphs are described from 10,000 with $\rho = 0.75$.

Table 5: Statistics of the hedge performances for each hedging strategies.

	MVH	Exp. Naive	Exp. $\gamma = 0.005$	Exp. $\gamma = 0.01$
Ave.	− 0.0743	− 0.4510	− 0.1809	0.0178
Std. Deviation	8.5671	8.8609	77.9953	8.8557
Min.	− 52.5175	− 43.2298	− 47.1811	− 42.8069
Max.	52.7244	40.5670	40.4875	40.9899
Success Hedge Ratio	55.68%	54.64%	56.50%	56.63%

performance is the exponential hedging with the utility indifference price, the success hedge ratio is 56.63%. The second best performance is the mean-variance hedging which hedge ratio is 55.68%. The exponential hedging with the mean-variance hedging cost gives the worst results.

5.2 Risk-Aversion and Hedging Performance of Exponential Hedging

Observing Table 2–5 makes us characteristics for the exponential hedging in the risk-management of view. From Table 2, the higher risk-averse requires more hedging cost which is used the utility indifference price. According to that, the exponential hedging successes to reduce the possibility of the shortfall (Table 3, 5). These results are interpreted as follows: The risk-aversion has role to control the shortfall risk in the exponential hedging. Elimination of the shortfall risk is accomplished via the utility indifference price. On the other hand, the standard deviation of the hedge error increases with the risk-aversion.

6 Concluding Remarks

In this paper, we implemented and discussed the hedging performance for the mean-variance hedging and the exponential hedging. We observe the relations between risk and return and between cost and performance from comparison of the mean-variance hedging and the exponential hedging. If one wants to reduce the shortfall risk, she/he must invest more money to the hedging. Also, if one wants to obtain the high success hedge ratio (i.e., hedge return), she/he should take more risk. We did not only compare two distinct hedging strategies, also demonstrated the hedging strategies with different initial hedging costs. When we use the mean-variance hedging cost as an ad-hoc way, the exponential hedging did not provide its advantages at all. On the other hand, using the utility indifference price as the initial cost implemented hedge performance. This alerts that the wrong estimation of the hedging cost gives bad hedging results. Finally, we compare the hedge results of the exponential hedging strategies for different risk-aversions. From this experimentation, we obtained the characteristics of the exponential hedging such that the risk-aversion manages the shortfall risk through the utility indifference price.

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