How can we handle too many criteria/alternatives?
: A study on AHP structural design

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Abstract

The pairwise comparison method (PCM) using a scale that indicates the importance of one element over another element with respect to given criteria/alternatives. In particular, in the field of Analytic Hierarchy Process (AHP), PCM has been used as the mainframe tool. Many different methods using a pairwise comparison have simple and high accuracy judgment. However, the number of pairwise comparisons will increase as the number of alternatives to be increased. Therefore, the burden of decision-makers becomes heavy. As a result, it can cause bad effects on the consistency. In this study, we propose a design method of the pairwise comparison matrix in consideration of the constraints of the decision makers, to reduce the effort of the task of the pairwise comparison and to improve the consistency of PCM. The results are demonstrated through a case of sport player choice problem.

Keywords: analytic hierarchy process, pairwise comparison matrix, experimental design

1. Introduction

Over the last several decades, a number of methods have been developed which use pairwise comparisons of the alternatives and criteria for solving multi-criteria decision-making problems. In the pairwise comparison method, criteria and alternatives are presented in pairs of one or more decision makers. It is necessary to evaluate individual alternatives, deriving weights for the criteria, constructing the overall rating of the alternatives and identifying the best one. It has been applied during the last years in many decision-making problems and has been used on a wide range of applications in many different fields. The method uses a reciprocal decision matrix obtained by pairwise comparisons so that the information is given in a linguistic form.

The Analytical Hierarchy Process (AHP) is a multi-objective decision-making method developed by Saaty[11–16]. It aims at quantifying relative priorities for a given set of alternatives on a ratio scale, based on the judgment of the decision-maker, and stresses the importance of the intuitive judgments of a decision-maker as well as the consistency of the comparison of alternatives in the decision-making process[11]. Since a decision-maker makes a decision based on knowledge and experience, the AHP approach depends on with the behavior of a decision-maker. The strength of this approach is that it organizes tangible and intangible factors in a systematic way, and provides a structured yet relatively simple solution to the decision-making problems[10]. Moreover, The Consistency Index (C. I.) is used to measure the reliability of the pairwise-comparison of AHP. A pairwise-comparison should be carried out carefully because all factors should be included, such as time cost. Satty consistency index C. I. is an AHP method, which applies typical criteria to check the reliability of the pairwise comparison matrix value. Shibayama, Nishina (1992)[9] and Satty

(2003)[16] have examined a number of numerical experiments, regarding the usefulness of C. I..

However, AHP uses a great number of AHP pairwise comparisons for calculating numerical weights of criteria and alternatives in general. While methods of using pairwise comparison have merits of simple process and accurate evaluation, many studies have indicated that decision maker's comparing burden is heavy if compared entities increase[18]. There exists a similar demerit in AHP because AHP needs to compare with many pairs of criteria if a number of criteria increases for more exact evaluation. Furthermore if a number of alternatives increases then it is necessary to compare with many pairs of alternatives for each criterion on a higher level. Many studies pointed out the heavy burden of pairwise comparison in AHP[17].

Therefore, in this study we concentrate on a problem of sharing the heavy burden of pairwise comparison work with multiple decision makers as possible under an assumption of making consensus building. The problem is transformed to group AHP problem and it needs a consensus building process to solve the problem. The results are demonstrated through a case of sports match.

2. AHP & CI

Based on pairwise comparison, Saaty proposes the analytic hierarchy process as a method for multi-criteria decision-making. It provides a way of breaking the general method top-down into a hierarchy of sub problems, which are easier to evaluate for the criteria/alternatives. Since its invention, AHP has been a tool available to decision-makers and researchers and is one of the most widely used multiple criteria decision-making tools (Vaidya and Kumar 2006)[7]. It is designed to cope with both the rational and the intuitive decision-making to select the best from a number of alternatives evaluated with respect to several criteria. In this process, the decision maker carries out simple pairwise comparison judgments, which are then used to develop overall priorities for ranking the alternatives (Saaty and Vargas 2001)[15]. The form of matrix of the pair-wise comparisons is as follows:

$$A = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_1/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix} \end{array}$$

The comparisons are made using a scale that indicates the importance of one element over another element with respect to a given attribute. Table 1 shows the scale ranges from 1 for 'the least valued than' to 9 for 'the most important than'.

Table 2.1. 1–9 Scale for the pair wise comparison (Saaty 2001)

Linguistic term	Preference number
Equally important	1
Weakly more important	3
Strongly more important	5
Very strong important	7
Absolutely more important	9
Intermediate values	2, 4, 6, 8

In the basic structure of an Analytic Hierarchy presented in Figure 2.1, the goal is specified at the top; all the objectives or criteria are listed below the goal and all alternatives are presented at the last level. Some key and basic steps involved in this methodology are:

Step 1. Determine the problem.
Step 2. Structure the decision hierarchy of different levels constituting goal, criteria, sub-criteria and alternatives.
Step 3. Compare each element at the related level and establish priorities.
Step 4. Perform calculations to find the normalized values for each criteria / alternative. Calculate the maximum Eigen value and C. I..
Step 5. If the maximum Eigen value, C. I. is satisfactory, then the decision is made based on the normalized values. If not, the procedure is repeated until the values lie in the desired range.

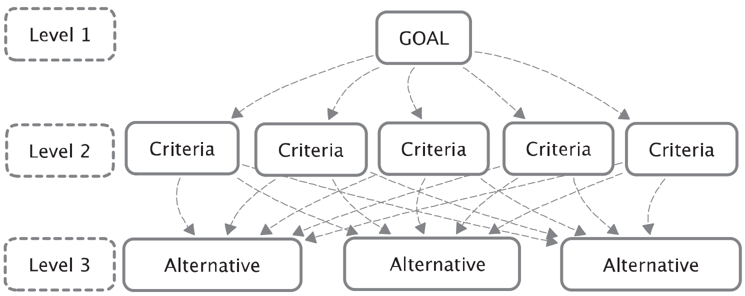

Figure 2.1. Basic structure of AHP

The consistency analysis is a part of the AHP method. It is to assure a certain quality level of the decision. The measure of inconsistency can be used to successively improve the consistency of judgments (Saaty and Vargas 2001)[15]. The formula 2 and 3 is generated to determine the convenience of the numerical judgment. In this respect, we calculated the C. I. confirming Saaty, which is defined as a ratio between the consistency of a given evaluation matrix and the consistency of a random matrix.

3. Consistency index & the number of criteria/alternatives

Once the model is built, the decision-makers evaluate the elements by making pairwise comparisons. When all the comparisons are completed, we calculate the priorities and the measure of consistency in our judgment. This is to assure the quality level of decision. The measure of inconsistency can be used to successively improve the consistency of decision-making (Saaty 2001) [15]. The equation below is generated to determine the convenience of the pairwise-comparison. The C. I. is not less than 0.10. In this respect, we calculated the C. I. confirming pairwise comparison to improve the reliability of AHP. C. I. is related to the eigenvalue method:

$$CI = \frac{\lambda_{\max} - n}{n-1},$$

where $\lambda_{\max}$ indicates maximum eigenvalue.

But, the C. I. is very sensitive to the task of pairwise-comparison by the decision maker. It takes significant costs (time, money) when the decision maker fills a pairwise-comparison matrix. A pairwise comparison is the process of comparing the relative importance, preference, or likelihood of two elements with respect to an element in the level above. A comparison is made with respect to each pair (the number of comparisons will be $\frac{n(n-1)}{2}$, where n is the number of criteria/alternatives in the model). Although the most advanced instruments are proposed, it is difficult to obtain consistency in practice, which makes it necessary to have a method capable of evaluating the importance of this precision to a specific problem. In this paper, we consider that inconsistency is a violation of proportionality, which can sometimes mean the violation of transitivity.

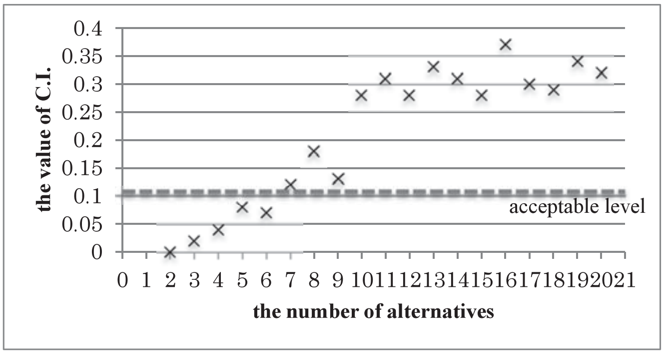

Figure 3.1. Variation of CI by increase in the number of alternatives

AHP utilizes a great number of pairwise comparisons for calculating numerical weights of criteria/alternatives. While the methods of using pairwise comparison have merits of simple process and accurate evaluation, many studies have indicated demerits that decision maker's comparing burden is heavy if comparing entities increase. We examined the relationship of criteria/alternatives increase and CI conversion in the target 50 people. As we can see from Fig. 3.1, the value of CI exceeds the acceptable range when the numbers of criteria/alternatives is more than 7. For proper number of criteria/alternatives in case of a single decision maker, concept of a magical number has been introduced as 7 ± 2, which was suggested by Muller (1956)[2]. After that, magical number of 5 to 9 has been changed to number of less than or equal to 4 in order to satisfy transitive law as possible in Cowan (2001)[8]. However, in the case of real selection problems, often the number of criteria/alternatives is more than five or ten, in some cases there may be 20 or more.

4. Pairwise comparison and its matrix design

In this section, we consider an athlete selection problem. There are 9 candidate players such as *A, B, C, D, E, F, G, H, I* which are evaluated by an examiner based on some criteria. In addition, we assume a physical constraint of examiner who can judge only three players. That is, one examiner can compare in three combinatorial pairs from 9 candidate players. The number of all pairs is ${}_9C_2 = 36$ from 9 candidate players. Because one examiner can test three pairs of all candidates, it is necessary to split all pairwise comparisons into 12 examiner. All candidate players are named by *A*, *B*, ..., *I* and only 3 candidate players of all candidates must be assigned to each examiners. How to assign all candidate players to 9 examiners fairly? The above problem has been studied in the design theory and some methods are proposed such as cyclic design method. As a result of utilizing cyclic design method, optimal design is obtained as follows;

{A, B, C}, {A, D, E}, {A, F, G}, {A, H, I}, {B, D, F}, {B, E, H}, {B, G, I}, {C, D, I}, {C, E, G}, {C, F, H}, {D, G, H}, {E, F, I}. In Table 4.1, we illustrate the above optimal design as matrix, which is called incidence matrix. Titles of rows and columns are candidate players and examiners in matrix, respectively. If there is design then 1 is assigned, otherwise 0. As a result, we obtain 9×12 matrix in Table 4.1.

Table 4.1 Incidence matrix of candidate player and examiner

	1	2	3	4	5	6	7	8	9	10	11	12
A	1	1	1	1	0	0	0	0	0	0	0	0
B	1	0	0	0	1	1	1	0	0	0	0	0
C	1	0	0	0	0	0	0	1	1	1	0	0
D	0	1	0	0	1	0	0	1	0	0	1	0
E	0	1	0	0	0	1	0	0	1	0	0	1
F	0	0	1	0	1	0	0	0	0	1	0	1
G	0	0	1	0	0	0	1	0	1	0	1	0
H	0	0	0	1	0	1	0	0	0	1	1	0
I	0	0	0	1	0	0	1	1	0	0	0	1

*. row: candidate players, column: examiners

In incidence matrix, there are three 1 in any column and it means that any examiner judges three candidate players. And there are four 1 in any row it means that any player is judged by three examiners. Furthermore, we know that the matrix reflects all pairs of wine and sommelier are compared.

Partially, the combinatorial design theory traces its origin to recreational mathematics in middle of 19th century. However, the theory was activated and developed by association with the design of experiments (DOE). It is the start of DOE that R. A. Fisher studied as an agricultural test in Rothamsted in 1919. He introduced terms of plot, treatment, block and variety to his experiment. These terms and related symbols due to R. A. Fisher are used as it stands in design theory. For example, symbol of treatment for each plot is used by $\boldsymbol{v}$ of denoting variety in DOE which was suggested by Fisher (1935)[10] and Ishii (1972)[3].

Now we consider representation of design by symbol. As mentioned before, number of treatments is denoted by $\boldsymbol{v}$. In the example of wine test, number of wine brands is denoted by $v = 7$ and number of sommeliers is $\boldsymbol{b} = 7$.

Table 4.2 Symbols in Players choice problem

	plot (k)	Treatment (v)	Block (b)	r	λ
Choice Players	Examiner's Constraints	Number of candidate players	Number of Examiners	Number of examiners on each player	One testing
	3	9	12	4	1

Next, $\boldsymbol{r}$ denotes number of blocks for each treatment and $\boldsymbol{r} = 4$ in wine test problem. Symbol $\boldsymbol{k}$ denotes a number of plots for each block and $\boldsymbol{k} = 3$ in wine test problem. Finally, $\boldsymbol{\lambda}$ denotes number of blocks in case there are any treatment pairs and $\boldsymbol{\lambda} = 1$ in wine test problem. (Table 3.2)

Under the above setting of notations, ($\boldsymbol{v}$, $\boldsymbol{b}$, $\boldsymbol{r}$, $\boldsymbol{k}$, $\boldsymbol{\lambda}$) are used to describe combinatorial design problems suggested by Mazur (2010). For instances, our problem is expressed by (9, 12, 4, 3, 1) design. The above design problem is efficiently utilized by sharing heavy burden of comparing work by multiple sommeliers for the AHP problem of a constrained physical limitation of a decision maker.

5. Fair sharing problem of burden

In this study, we suppose that decision maker can cover only 3 entities because of physical limitation. The assumption is based on a minimum number of entities for evaluation exactly and there are 3 combinatorial pairwise comparisons. For example, if wines of $\{A, B, C\}$ are assigned then decision maker tests three pairwise comparisons of (A, B), (A, C) and (B, C). However, there is no meaning to share comparison work when decision maker tests only two wines of {A, B} because there is only one pairwise comparison.

Table 4.3 Symbols in Players choice problem

all pairwise comparisons							
AB	*BC*	*CD*	*DE*	*EF*	*FG*	*GH*	*HI*
AC	*BD*	*CE*	*DF*	*EG*	*FH*	*GI*	
AD	*BE*	*CF*	*DG*	*EH*	*FI*		
AE	*BF*	*CG*	*DH*	*EI*			
AF	*BG*	*CH*	*DI*				
AG	*BH*	*CI*					
AH	*BI*						
AI							

Allocation design
examiner −01: $\{A, B, C\}$
examiner −02: $\{A, D, E\}$
examiner −03: $\{A, F, G\}$
examiner −04: $\{A, H, I\}$
examiner −05: $\{B, D, F\}$
examiner −06: $\{B, E, H\}$
examiner −07: $\{B, G, I\}$
examiner −08: $\{C, D, I\}$
examiner −09: $\{C, E, G\}$
examiner −10: $\{C, F, H\}$
examiner −11: $\{D, G, H\}$
examiner −12: $\{E, F, I\}$

Table 4.3 considers wine test problem with 9 candidate players and 3 physical limitations. There are 36 pairwise comparisons for each wine and design of assigning 3 players to 12 examiners. In short, sommelier 1 judges 3 players of {A, B, C} and compares 3 pairs of (A, B), (B, C) and (A, C). If there are 8 candidate players then all pairwise comparisons of ${}_8C_2 = \frac{8 \times 7}{1 \times 2} = 28$ are made and necessary number of sommeliers is obtained $28/3 = 9.33....$ from the assumption of physical limitation. Therefore, there is no optimal design because the number of sommeliers is not positive integer.

In Table 4.4 if number of players is given then number of all pairwise comparisons is obtained. Next, if we divide the number of all pairs by number of physical limitations then number of necessary sommeliers is obtained. Column 6 denotes judgment 1 (J1) of checking the existence of positive integer of necessary number of sommeliers.

Next, number of all pairwise comparisons is ***bk*** because ***b*** sommeliers taste ***k*** wines. On the other hand, number of all pairwise comparison is ***vr*** because *v* wines are assigned to ***r*** sommeliers. Therefore, it holds equality of $\boldsymbol{bk} = \boldsymbol{vr}$ and it is a necessary condition to the existence of design. The equality is used as judgment 2 in column 7 on table 4.4. J2 find positive integer from number of examiners for each player. It is necessary to satisfy all judgments 1, 2 simultaneously for fair sharing with comparison works. In Table 4.4, we list up the optimal design by cyclic method after judging the existence of design. For detailed cyclic method, refer to note of Rosa (1991)[1].

Table 4.4 Judgment of existence of design

the number of pairwise comparison (t)	the required number of examiners (b)	Constraints of examiners (k)	the number of players (v)	the number of appearance blocks of each players (r)	Judgement-1 (Integer)	Judgement-2 (Fairness)	Final Judgment
3	1	3	3	1	Pass	Pass	◎
6	2	3	4	1.5	Pass	Failure	X
10	3.333333333	3	5	2	Failure	Pass	X
15	5	3	6	2.5	Pass	Failure	X
21	7	3	7	3	Pass	Pass	◎
28	9.333333333	3	8	3.5	Failure	Failure	X
36	12	3	9	4	Pass	Pass	◎
45	15	3	10	4.5	Pass	Failure	X
55	18.33333333	3	11	5	Failure	Pass	X
66	22	3	12	5.5	Pass	Failure	X
78	26	3	13	6	Pass	Pass	◎
91	30.33333333	3	14	6.5	Failure	Failure	X
105	35	3	15	7	Pass	Pass	◎
120	40	3	16	7.5	Pass	Failure	X
136	45.33333333	3	17	8	Failure	Pass	X
153	51	3	18	8.5	Pass	Failure	X
171	57	3	19	9	Pass	Pass	◎
190	63.33333333	3	20	9.5	Failure	Failure	X

In this study we consider an AHP problem to judge candidate players. The AHP problem is illustrated by AHP chart in Figure 5.1. For simplicity, we suppose that there are players of less than 10 and one decision maker (examiners) has physical limitation of 3 players for judging. For this problem, we propose an efficient algorithm to share heavy testing burden fairly with multiple examiners (sub layer) based on the combinatorial design theory with cycling method. It is necessary to investigate the existence of optimal design and construction of the design for fair burden sharing. The above AHP problem is revised by add sub- layer such as Figure 5.1.

Solution procedure of fair sharing problem

Step1: Set the test problem to classic AHP chart.
Step2: Define a physical limitation of decision maker.
Step3: Find necessary number of sub decision makers (the number of examiners in our case).
Step4: Find optimal design assigning each wine to each sub decision maker, fairly. (Experimental Design method & Cycling method).
Step5: Revise the classic AHP chart by adding layer of sub decision makers.
Step6: Solve the problem by Group AHP (merging).

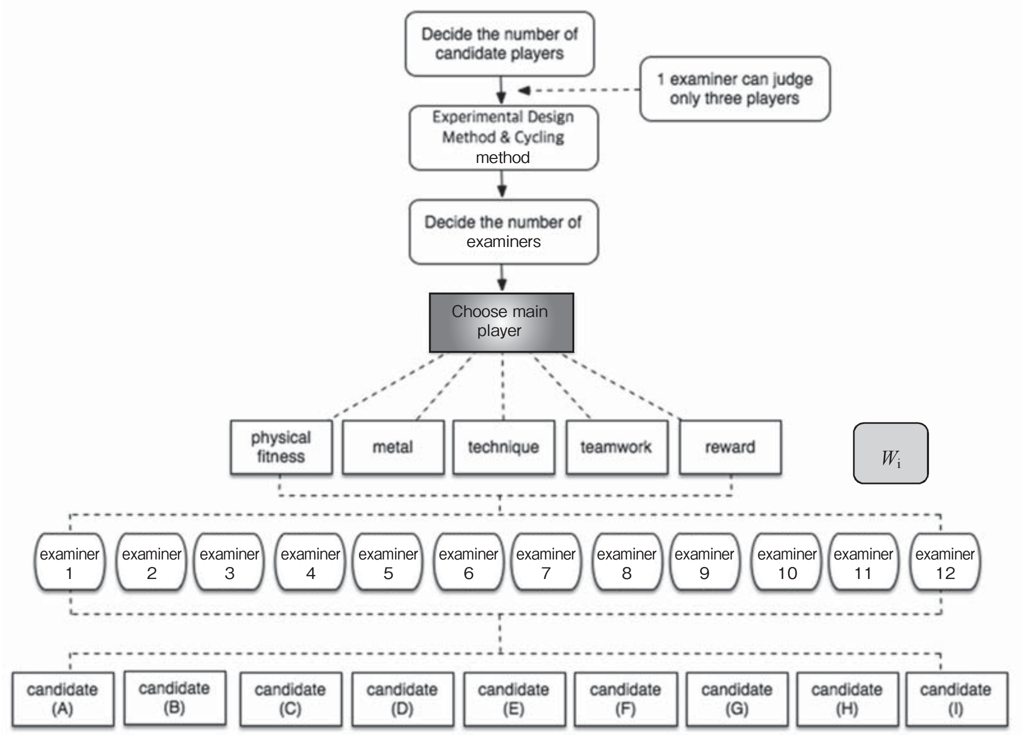

Figure 5.1. Hierarchy chart for sharing burden

W_i implies the weight of each criterion and they have the same weight because of the examiners have to evaluate the same scale. And then, each examiner evaluates allocated candidate players in Table4.3. Finally, this hierarchy model needs to merge process like in our model in Figure5.1. For the above solution procedure, computational complexity and validity are clearly based on the theory of combinatorial design and AHP.

6. Dscussion and Conclusion

In this study, we proposed an AHP problem with physical constrains, such as too many criteria/alternatives. For sharing the heavy burden of pairwise comparison work, the problem is transformed to group AHP problem with multiple decision makers. In the transformation, we proposed a proper number of multiple decision makers for physical constraints and fair sharing of burden based on combinatorial design theory using experimental design method and cycling algorithm. For simplicity, we treated the case as a problem with limited physical constrains and entities. In future, more research is necessary to discuss more theoretical issues.

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